Preface for Students and Instructors

Math textbooks are usually heavy behemoths. They try to provide every argument and example ever wanted. In other words, they hope to be used like an encyclopedia during and after the course. This textbook is gloriously lightweight, even when printed, which is possible because this book has different goals from a traditional book. We hope this text will guide its users to develop the skills, attitudes, and habits of mind of a mathematician.

Of course, it only makes sense to hire a guide if you've got some new territory to explore. Users of this book will become active explorers—experiencing mathematical thinking on their own. What equipment should you bring with you on this exploration? In short, all you need is the desire to work hard and an interest in thinking more clearly. You are not expected to have any previous experience with mathematical proof. If you have some exposure to calculus, then you will already have some experience with mathematical abstraction that will support your exploration, but the content of a calculus course is not needed for this adventure.

This book encourages its users to behave as much like practicing mathematicians as possible. We want students and other people who use this book to learn and to enjoy the process of distilling and exploring ideas. This book helps to embed habits of inquiry in users through the exploration of interesting mathematical content. The book presents a carefully designed sequence of exercises and theorem statements so that its users will be guided to discover both mathematical ideas and also strategies of proofs and strategies of thinking. This preface to students and to instructors explains how the book might be used.

For Students. Mathematics is not a spectator sport. To learn mathematics deeply, it is important to actively grapple with each new idea, putting the pieces into context and seeing connections and meaning. This book presents you with a sequence of challenges that are designed to give you the experience of developing mathematics for yourself. During your work with this book, you will become increasingly proficient at proving theorems on your own and at learning new ideas on your own. You will find that you gain confidence and competence in facing challenging issues.

The strategy of presentation in this book may be a bit unusual to you. You may wonder why we didn't simply write the proofs of the theorems in the book, rather than omitting the proofs and expecting you to figure them out for yourself. Researchers in the field of human learning have found that the best learning occurs through a process of engaged struggle with the ideas. In mathematics, this means constructing proofs. The struggle can be enjoyable along the way, but personally working through ideas lies at the heart of how to learn. And learning how to learn is the real goal of an education.

At first you may be uncertain about what constitutes a mathematical proof and how to even begin to prove something on your own, but after a short while, you will find that you have mental strengths that you did not have before. Like a beginner at tennis, at first it seems impossible to imagine how to strike the ball, much less direct it to where you want it to go. Watching videos of expert tennis players or hearing explanations can only help so much. The basic part of learning a skill is in doing it yourself, including those early steps during which you will inevitably make many mistakes. Just be confident that through your own attempts at doing the exercises and proving the theorems, you will find yourself becoming better and better.

At the very heart of this book is the assumption that you can and will be asked to think about many questions *before* a teacher or classmate tells you how to think about them. This process can be uncomfortable for many students, but it is absolutely critical for making the transition from consumer of knowledge to producer of knowledge that is the goal of a college education. In general, you should read the text up to the point of an exercise or theorem and then work on that question, only reading more when you are finished. Many students have had this experience before you, and the exercises and theorem statements are designed to present challenges that will guide you toward increased independent mathematical ability. Try not to get discouraged in the early days when you might well feel a bit confused and overwhelmed. Instead, confidently keep trying and you will become a better thinker.

For Instructors. This book is intended to be used in support of a guided discovery, Inquiry Based Learning method of instruction. When we use the book in our own classes, we ask the students, working either individually or in groups, to do the exercises and prove the theorems and then to present their work to the class; a substantial amount of this work is done by the students in preparation for class. Then we involve the other students by asking them to review part of a proof or to otherwise comment on the presentation. This format of class activity soon gives students the idea that they need to think through ideas on their own. Once they firmly accept that they can personally explore the unknown and can personally determine whether an argument is correct or not, then whole worlds open up to them. They become producers of knowledge rather than consumers of knowledge. Their standards about what they view as understanding rise in this class and other classes; understanding comes to include the ability to explain an idea.

Recent educational research has shown that an early inquiry-based learning experience is particularly beneficial. These kinds of experiences may seem as though they would only work for strong students, but the research also shows that inquiry-based learning experiences are *more* beneficial on average for previously low-achieving students.

If you do not have experience teaching using an Inquiry Based Learning approach, there are many materials about Inquiry Based Learning instruction available, including a mentoring network and workshops. The website for the Academy of Inquiry Based Learning, www.inquirybasedlearning.org, can direct you to many helpful resources. There are many strategies for conducting an IBL course, and the AIBL website can help you to find a method that fits your needs.

The *Through Inquiry* collection of resources is intended to provide instructors with the flexibility to create textbooks supporting a whole range of different courses to fit a variety of instructional needs. To date, the series has four e-units treating, graphs, groups, ε-δ calculus, and number theory. You may combine any collection of these units to create a textbook or textbooks tailored to your needs. Here are some brief descriptions of these units.

- Graphs: This unit asks users to explore the famous ideas flowing from the Königsberg bridge problem, planarity, and colorability. This unit is concrete and visual, which supports students who are new to mathematical proof and abstraction; however, some students find it challenging to be explicit about ideas that are visually obvious to them. It contains several situations that encourage students to generate algorithmic constructions; this experience is particularly effective at getting students to be explicit about their ideas. The theorems in this unit have several applications, which can help motivate those students who are attracted to the usefulness of mathematics.

- Groups: This unit asks users to explore the fundamental algebraic structures of addition, multiplication, and symmetry composition towards a partial classification of groups. This unit is "categorical" in the sense that it leads students to explore sub-objects, product objects, quotient objects, et cetera; as a result, this exploration provides the most clear and explicit list of automatic responses that mathematicians have to new definitions. This unit is quite abstract. It is particularly good at helping students learn to return to the precise language of the definitions. This unit is particularly appealing to those students who love the rigor and abstract beauty of mathematics.

- Calculus: This unit asks users to explore the rigor behind limits, continuity, derivatives, and integrals in ε-δ calculus. This unit contains calculus content that will probably be familiar to students, which is both helpful and challenging. The students who know calculus will bring intuition about the results, which will allow them to focus on the argumentation; however, the same students are also more likely to try to recall vague explanations from previous courses instead of generating and evaluating new proofs. We believe this unit could also be used with students who do not know any calculus, though it should be taken at a much slower pace, and the students should be asked to work with more examples than are included here. The definitions in this unit contain the long strings of quantifiers that make this topic especially challenging. This unit is particularly appealing to students who intend to teach calculus and to students who seek to understand the mathematical basis of a familiar topic.

- Number Theory: Number theory is a great vehicle for an introduction to proof course. Students are familiar with basic properties of numbers, and yet the further study of number theory leads to fascinating and extremely deep mathematics. This introduction to number theory starts from familiar concepts about integers and leads students to discover the strategies of using definitions, exploring examples, and proving theorems. It treats modular arithmetic, primes, RSA encryption, and additional topics. This extensive unit contains sufficient material for a one semester course plus additional topics for independent study.

Inquiry-based instruction can be used in courses with several different goals. It can be used in introduction-to-proof courses. It can be used to build an alternative entry to the mathematics major, giving students a view of mathematics different from what they have probably seen before, a little more like research. It can be used in a course that gives students who may not be going on in mathematics a sense of some abstract mathematics besides calculus. It is especially valuable for future or current teachers. It is ideal for independent study. And it can be used in more traditional courses to emphasize student-centered learning.

This collection of units contains sufficient material for several semesters of courses, so you will have to choose which units will serve your students for each course. You may choose to select only some of the exercises and theorems in a given unit, particularly if you choose to do more than one unit in a semester. No matter what units you choose and approach you take, you should always leave room for students to ask their own, additional mathematical questions and to state, explore, and prove their own conjectures. Sample, specific threads are described in the Instructor's Resource; however, the main principle is to modify your decisions based on your own students' needs.

Here are some suggestions for which units to use for various purposes.

- **Introduction to proof** – Courses intended to introduce students to proof need to include many opportunities for the students to construct proofs; inquiry-based courses are ideal for providing those opportunities. You could choose to spend the entire term on one topic. Number theory contains enough material for a one semester course; the graph, group, and calculus units would need to be done in great depth or augmented with additional challenges if you choose to treat only one of these topics. Instead, you could choose two or three of these units. We think that graph theory and number theory are particularly good topics for introducing proofs. One advantage of choosing multiple topics for this course is that students get to start fresh with new definitions and see the progress they have made in writing proofs. None of these options assumes that the students have been taught about proof structures. When we use these units for an introduction to proof course, the common proof techniques are distilled through discussion and reflection from the arguments generated by the students.

- **Courses for pre-service/in-service teachers** – An inquiry-based course is particularly important for teachers, because it models a strategy of instructional design; it gives them practice communicating mathematics; it allows them to see the learning process of other students; and it encourages deep understanding of the material that leads to a stable and nuanced understanding. Teachers working with younger students would benefit from a course that uses number theory because numbers are the central concept of elementary school mathematics. Teachers working with high school students need to understand the basics of abstract algebra and real analysis in order to provide coherence to the high school mathematics. So they would benefit from a course that uses the group theory and calculus units.

- **Alternative entry to the major** – The traditional sequence of mathematics courses for mathematics majors has the unfortunate drawback of avoiding topics in abstract mathematics beyond calculus until later in the students' careers. An alternative is to create a non calculus-based introduction available to students as early as the first year. Selecting topics from the graph, group, and calculus units could provide enticement to further study of mathematics that gives them a taste for a variety of different flavors of mathematics. Such a course is also ideal for science students who would otherwise not see mathematics beyond calculus.
- **Survey for applied math majors** –All mathematics majors should have a working understanding of the basic ideas of algebra and analysis. A course that uses the group theory and calculus units could serve as a survey course treating these two topics.
- **Independent study** – Any unit or collection of units could be used as a resource for a single user or a small group of users who want to learn independently about mathematics. This could be in the form of an independent study under the guidance of a mathematician or a self-guided experience. Since there are many challenging exercises and theorems, learners can do a great deal of work independently. Working through the units provides an excellent quasi-research experience.
- **Core/Topics Courses** – Inquiry-based learning is an effective way for students to learn mathematics. These units represent core and elective topics common in many mathematics departments. An inquiry-based learning experience foster habits of mind, like inquisitiveness and self-efficacy, that are important for all students.

We hope that all users of this book will find it a rich source of enjoyable challenges.

Acknowledgments

We thank the Educational Advancement Foundation and Harry Lucas, Jr. for their generous support of the Inquiry Based Learning Project in general and their support of our work on this project in particular. The Inquiry Based Learning Project has inspired us and has inspired many other faculty members and students across the country. The Educational Advancement Foundation has a clear purpose of fostering methods of teaching that promote independent thinking and student creativity. We hope that this project will contribute to making inquiry based learning methods of instruction broadly available to many faculty members and students nationally. We thank Chris Thoren for his work on some of the images. We also thank the National Science Foundation for its support of this project under NSF-DUE-CCLI grants 0536839 and 0920173. In addition, we would like to thank the Department of Mathematics at The University of Texas at Austin and the Department of Mathematics and Computer Science at Augustana College for creating opportunities in which this work could be carried out.

We would like to thank everyone who has helped this project evolve and become more refined. Most importantly, we would like to thank the many students who have used versions of these units in our classes. They helped us to make the presentation more effective and they energized us by rising to the challenges of personal responsibility, rigor, and curiosity. They inspired us to aim high in education–to realize that mathematics classes can genuinely help students to become better, more creative, more independent thinkers.