Preface

Calculus is a collection of incredibly efficient and effective tools for handling a wide variety of mathematical problems. Underpinning these methods is an intricate structure of ideas and arguments; in its fullest form this structure goes by the name of *real analysis*. The present volume is an introduction to the elementary parts of this structure, for the reader with some exposure to the techniques of calculus who would like to revisit the subject from a more conceptual point of view.

This book was initially developed for the first semester of the Honors Calculus sequence at Tufts, which gives entering students with a strong background in high school calculus the opportunity to cover the topics of the three-semester mainstream calculus sequence in one year. It could, however, prove equally useful for a course using analysis as the topic for a transition to higher mathematics.

The challenge of the Honors Calculus course was twofold. On one hand, the great variability among high school calculus courses required us to cover all the standard topics again, to make sure students end up "on the same page" as their counterparts in the mainstream course. On the other hand, these bright students would be bored out of their minds by a straight repetition of familiar topics.

Keeping in mind these competing demands, I opted for a course that approaches the tools of calculus through the eyes of a mathematician. In contrast to "Calculus Lite", the present book is "Calculus Tight": a review of often familiar techniques is presented in the spirit of mathematical *rigor* (hopefully without the *mortis*), in a context of ideas, and with some sense of their history. For myself as a mathematician, the intellectual exercise of working through all the theorems of single-variable calculus and constructing an elementary but coherent and self-contained presentation of this theory has been a fascinating experience. In the process, I have also become fascinated with the history of the subject, and hope that my enthusiasm will help to brighten the exposition.

Idiosyncracies

After three centuries of calculus texts, it is probably impossible to offer something truly new. However, here are some of the choices I have made:

Level: "Calculus" and "analysis"—which for Newton *et al.* were aspects of the same subject—have in our curriculum become separate countries. I have stationed this book firmly on their common border. On one side, I have discussed at some length topics that would be taken for granted in many analysis courses, in order to fill any "gaps" in the reader's previous mathematical experience, which I assume to be limited to various kinds of calculation. I have also limited myself to mathematical topics

associated with the basics of calculus: advanced notions such as connectedness or compactness, metric spaces and function spaces demand a level of sophistication which are not expected of the reader, although I would like to think that this book will lead to attaining this level. On the other side, I have challenged—and hopefully intrigued—the reader by delving into topics and arguments that might be regarded as too sophisticated for a calculus course. Many of these appear in optional sections at the end of most chapters, and in the *Challenge Problems* and *Historical Notes* sections of the problem sets. Even if some of them go over the heads of the audience, they provide glimpses of where some of the basic ideas and concerns of calculus are headed.

Logic: An underlying theme of this book is the way the subject hangs together on the basis of mathematical arguments. The approach is relatively informal. The reader is assumed to have little if any previous experience with reading definitions and theorems or reading and writing proofs, and is encouraged to learn these skills through practice. Starting from the basic, familiar properties of real numbers (along with a formulation of the Completeness Property)—rather than a formal set of axioms—we develop a feeling for the language of mathematics and methods of proof in the concrete context of examples, rather than through any formal study of logic and set theory. The level of mathematical sophistication, and the demands on the proof-writing skills of the reader, increase as the exposition proceeds. By the end of the book, the reader should have attained a certain level of competence, fluency and self-confidence in using the rhetorical devices of mathematics. A brief appendix at the end of the book (Appendix A) reviews some basic methods of proof: this is intended as a reference to supplement the discussions in the text with an overview.

Limits: The limit idea at the heart of this treatment is the limit of a *sequence*, rather than the standard $\varepsilon - \delta$ limit of a function (which is presented in the optional § 3.7). At the level of ideas, this concept is much easier to absorb, and almost everything we need can be formulated naturally using sequences.

Logarithms and exponentials: Most rigorous treatments of calculus define the logarithm as an integral and the exponential as its inverse. I have chosen the ahistorical but more natural route of starting with natural powers to define exponentials and then defining logarithms as their inverses. The most difficult step in this "early transcendentals" approach, from a rigor point of view, is the differentiability of exponentials; see § 4.3.

History: I have injected some of the history of various ideas, in some narrative at the start of each chapter and in exercises denoted *History Notes*, which work through specific arguments from the initiators of the subject. I make no claims for this as a historical work; much of the discussion is based on secondary sources, although where I have had easy access to the originals I have tried to consult them as well.

Technology: There are no specific references to technology in the text; in my class, the primary tools are paper and pencil (or blackboard and chalk). This does not, of course, preclude the use of technology in conjunction with this book. I would, in

fact, love to hear from users who have managed to incorporate computer or graphing calculator uses into a course based on the present exposition.

How to use this book

Clearly, it is impossible to cover every proof of every result (or anything near it) in class. Nevertheless, I have tried to give the reader the resources to see how everything is supported by careful arguments, and to see how these arguments work. In this sense, the book is part textbook, part background reference. The beginning of each chapter sketches the highlights of the topics and tools to come from a historical perspective. The text itself is designed to be read, often with paper and pencil in hand, balancing techniques and intuition with theorems and their proofs.

The exercises come in four flavors:

Practice problems are meant as drill in techniques and intuitive exploration of ideas.

Theory problems generally involve some proofs, either to elaborate on a comment in the text or to employ arguments introduced in the text in a "hands on" manner.

Challenge problems require more ingenuity or persistence; in my class they are optional, extra-credit problems.

Historical notes are hybrids of exposition and exercise, designed to aid an active exploration of specific arguments from some of the originators of the field.

Of course, my actual homework assignments constitute a selection from the exercises given in the book.

A note at the start of each set of exercises indicates problems for which answers have been provided in Appendix B; these are almost exclusively Practice problems. Solutions to *all* problems, including proofs, are given in the **Solution Manual** accompanying this text.

On average, one section of text corresponds to one hour in the classroom. The last section of each chapter after the first is optional (and usually skipped in my class), although the notion of uniform continuity from § 3.7 appears in the rigorous proof of integrability of continuous functions.

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Finally, a word about Marty Guterman, to whose memory this work is dedicated. During the thirty or so years that we were colleagues at Tufts, and especially in the course of our joint textbook projects, I learned a great deal from this extraordinary teacher. His untimely death occurred before the present project got underway, but we had frequently discussed some of the ideas that have been incorporated in this text. His reactions and comments would have been highly valued, and he is missed.

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