Tomorrow's Geometry

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Introduction

In *A Century of Mathematics in America (Part II)*, Robert Osserman contributed an article entitled "The Geometry Renaissance in America: 1938-1988." The renaissance in geometry that he recounts has not been restricted only to America and did not end in 1988. Nor, as Osserman notes in a "postscript" to the article, has this renaissance been confined only to the area of differential geometry, which is what his article deals with. There are many reasons for the renaissance in geometry, ranging from new developments in physics and biology, to the development of the digital computer, to the flowering of old and new fields within mathematics that stimulate geometric insights.

Yet in terms of the way geometry is represented in the undergraduate curriculum, there has been no renaissance. On the contrary, there has been relatively little change in the nature of the geometric mathematics taught in colleges. This unchanging curriculum for geometry in college has in turn prevented any new geometry or applications stemming from geometry from being presented in high school (or earlier grades). Students planning to become high school or middle-school teachers usually see only that geometry covered in a survey course on Euclidean and non-Euclidean geometry. The content of such courses rarely discusses geometry that reaches into the period of which Osserman speaks. However, because of its unique intuitive and quick-starting nature, geometry that has been discovered in the last 30 years can be taught to undergraduates.

In light of these contrasting phenomena—explosive growth in geometric methods and applications of geometry but an unchanging undergraduate geometry curriculum—the MAA Curriculum Action Project convened an E-mail Focus Group to debate geometry education. A diverse group of research geometers and geometry educators agreed to participate in this electronic debate.

Questions To Grow On

In order to help participants in the electronic mail discussion think about the issues, key questions were formulated as a framework within which an informed exchange of views on educational reform for geometry could take place. In this introduction, each of the questions is followed by discussion that motivates the choice of the question as one of those to frame the debate. Although this background information appears here, only the questions were posed prior to the original electronic mail debate.

Question 1: What steps do you feel should be taken to improve the quality of undergraduate education in geometry? For example, what new courses might be developed? What changes are needed in existing courses? Are there suitable ways to include geometric topics in courses not specifically about geometry?

Background: Most undergraduate courses dealing with geometry belong to one of two types. One type of course, often designed for students bound for graduate school, is devoted to some specific part of geometric mathematics. Examples of such courses include Differential Geometry, Convexity and Geometric Inequalities, Theory of Graphs, Topology, Combinatorial Geometry, or Knot Theory. These courses (especially at smaller colleges) have often been developed by a member of the faculty with a research interest in the area, and are often taught exclusively by one faculty member. Such courses sometimes fall into disuse when this faculty member moves on to another college (or retires).

The other type of course will be referred to, with perhaps a slight abuse of language, as a "survey" course. These courses are sometimes taken by students bound for graduate school, but the most common type of student in such classes are prospective high school teachers. These courses are taught under such titles as Advanced Euclidean Geometry, Modern Geometry, Non-Euclidean Geometry, Projective Geometry, etc. Another common course title, often aimed at high school teachers and often having "survey" overtones, is Geometric Transformations.

Although courses of the first type often deal with mathematics developed since World War II, courses of the second type almost never do. The first type of course often provides depth but not too much breadth. The second type of course provides greater breadth, together with some depth, but is of little research interest to contemporary geometers. Both types of courses rarely mention applications of geometry external to mathematics.

Geometry enters the curriculum not only through courses specifically designed for their geometric content, but also in other courses which may either be required or elective for a mathematics major. Examples include geometric reasoning behind the theory of relative maxima and minima (and points of inflection) in calculus, and symmetry of polygons, polyhedra, and tilings when studying group theory in a modern algebra course. Yet it is not clear that conscious efforts are being made by textbook authors and course designers to systematically exploit the power of geometric reasoning and phenomena in non-geometry courses. In particular, geometry is so under-represented in terms of content and methodology that typical practitioners of mathematics may lack sufficient background to draw on a reservoir of geometric materials for their teaching.

Question 2: What strategies can be employed to get more modern geometric topics into the undergraduate curriculum, particularly for students who may only take introductory level courses?

Background: Most students who take courses designed for mathematics majors in their freshman and sophomore years do not in fact major in mathematics. Yet such students often enter fields in which mathematics plays a significant role (e.g., physics, engineering, and computer science). More often than not, these students take the calculus sequence and linear algebra but are unlikely to take a course with a high density of geometry.

As recently as the late 1950s and early 1960s it was not uncommon for students to precede their taking of calculus with a course in Analytic Geometry. Not only is this no longer true, but increasingly less and less attention to geometric ideas is being given within calculus courses. This is unfortunate, since there are many geometric approaches to calculus that afford insights into such fundamental ideas of calculus as limits, the derivative, area, arc length, etc. Traditional analytic geometry is hardly modern in flavor and what little is left

of this subject in current calculus courses is rarely forward-looking. One avenue that has been suggested to get more modern geometric ideas into calculus is via applications. For example, the field of robotics is a very active one and includes many questions of a geometric kind. (Examples include geometric questions arising from motion-planning problems, from questions concerning recognizing objects from various types of probes, and from geometric algorithms involved in questions concerning grasp and motion of robot arms and fingers.) Other fields that might prove to be fertile sources of ideas for bringing geometry into calculus are image processing, data compression, geometric modelling, and computer vision.

Linear algebra has, in the hands of some teachers, proved to be a more fertile area to introduce geometry. Geometric transformations, robots, linear programming, computer vision, and graphics are all subjects rich in geometrical content in which linear algebra is intimately involved. Graph theory also has many important interfaces with linear algebra, for example, in the study of Markov chains. Unfortunately, the introduction of geometric material, though it is closer to the surface here than in some other branches of mathematics, is not the norm. In fact, although algebra and geometry are handmaidens, the geometry behind and connected with linear algebra is often ignored.

However, a success story for modern geometric ideas of a specialized kind has been the creation of a number of courses in discrete mathematics for both mathematics majors and computer science majors. These courses often include extensive studies of the theory of graphs, of geometric algorithms (including in some cases topics from the fast growing area of computational geometry), and of cellular automata. Ironically, this course has been perhaps more successful in introducing computer scientists to geometry and geometric thinking since for them it is usually a required course. Some mathematics departments still refuse to grant credit to mathematics majors for discrete mathematics courses. Often discrete mathematics is a freshman course for computer science majors, which means that the geometric ideas they are exposed to becomes an early tool for them to exploit in later courses. For mathematics majors, any exposure they have to geometry is likely to be deductive geometry, taken at a much later point in their training.

Question 3: As a result of the recent recommendations regarding the preparation of teachers of school mathematics, what specific preparation in geometry is needed by prospective and practicing teachers?

Background: High school teacher preparation and the up-grading of geometric skills and knowledge among currently practicing high school teachers raises some especially difficult problems. Historically, the high school curriculum in the 10th grade has concentrated on the teaching of deductive Euclidean geometry with greatly varying degrees of rigor. So-called two-column proofs of Euclidean theorems have often been the goal in "elite" high schools. However, over the last ten years there has been a growing interest in teaching geometry at least in part in an inductive/problem-posing environment with support from educational technology. There are indications that many high school teachers teach Euclidean geometry as the geometry of the physical universe, and many teachers are unaware of the 19th-century discovery of non-Euclidean (Lobachevsky) geometry.

To give future teachers tools for teaching the existing curriculum, many colleges require high school teachers to take what is referred to as a "survey course" in geometry. These courses aim to show prospective high school teachers a unified framework for some of the

results taught in the traditional high school course (e.g., Ceva's theorem as a way to understand the concurrence of the medians, altitudes, and angle bisectors in a triangle) and to introduce them to the fact that other geometries than Euclid's can be consistent. Little time is left over to show future high school teachers such topics as tilings, polyhedra, box packings, etc., which are Euclidean in framework but much more modern in flavor than the circle, triangle, and quadrilateral geometry currently emphasized to the exclusion of nearly every other aspect of geometry.

Furthermore, few high school teachers are prepared, as a consequence of the geometry they learned, to alert high school students who show early interest in research in mathematics to anything other than a distorted view of the nature of research in geometry. Westinghouse projects and mathematics fair papers too often deal with the geometry of triangles, circles, and quadrilaterals which no longer have virtually any interest to geometers. Yet such quick-starting geometric areas as tilings of the plane, graph theory, polyhedra, and convexity are denied an army of bright assistant investigators. The Pascal triangle still often reigns supreme where, say, the study of polyominoes offers an equally rich array of lessons for young students. If teachers are not exposed to these concepts, they will be unable to pass such ideas along to their students.

Question 4: What role can computers and other new technologies play in improving the quality of undergraduate education in geometry?

Background: Over the last few years a variety of computer graphics tools, geometric tool kits, and computer languages with geometric primitives have been developed. Examples include large symbolic manipulation packages (such as *Maple, Derive,* and *Mathematica*) with elaborate two- and three-dimensional graphing techniques; specialized graphing packages designed to graph functions and surfaces; geometric tool kits for Euclidean geometry and geometric transformations (such as the "Geometric Supposer" and the "Geometer's Sketchpad"); computer environments to study graph theory (such as "Netpad," developed by Bellcore); CAD/CAM systems; and the computer language *Logo*.

All of these tools have profound implications for both research and teaching in geometry. These educational technologies make it possible to display and conduct calculations on geometric objects that would either be impossible or very time consuming with paper-and-pencil tools. Although some schools have been experimenting with using these technological tools in lower grades (e.g., *Logo* is introduced in some school systems as early as the fourth grade) and in high school (e.g., the "Geometric Supposer" and the "Geometer's Sketchpad"), most of these tools have remained largely experimental, used in industry but having little effect in classrooms. Other new technologies with research or pedagogical implications are videotape and videodisk materials with a geometrical theme and the availability of supercomputers to solve calculation-intensive geometric problems.

One particular problem with new technologies has been that their evolution has been so fast that college educators are often reluctant to promote any particular one of these software packages, languages, or environments for fear that what they are teaching will all too quickly become obsolete or outmoded. Yet many of these tools have grown and improved as they have matured. A good example is perhaps provided by the language *Logo*. Pioneered at MIT as a means for introducing elementary school children to geometric concepts and

exploratory environments, this language quickly became mired in innumerable different dialects (even for the same computer), thereby limiting the tremendous opportunities implicit in this marvelous tool. Geometers (who must share some of the blame here) have been very slow in showing how to use *Logo* in a creative manner (e.g., *Logo* can do more than draw pictures quickly of families of polygons). This has been in part due to fear that the next generation of tools would supersede the current one.

An example of how technology has interfaced with both pedagogical and research aspects of geometry is afforded by the role of computer graphics in the explosion of interest in fractals and dynamical systems. Without computers, the intriguing numerical and visual patterns that have stimulated large amounts of interest among students (and the lay public), not to mention an exponentially growing literature on chaos and related phenomena, would not have been possible.

Question 5: How can more interest in geometry be generated among undergraduate majors in the mathematical sciences?

Background: The traditional first year graduate courses in mathematics are Analysis, Algebra, and Topology. (Introductory topology is rarely taught primarily for its geometric content, but more often as a handmaiden for analysis.) The structure of qualifier and preliminary examinations in universities encourages students to select one of these fields in which to write a thesis rather than such smaller fields as Applied Mathematics, Logic, Number Theory, or Geometry. The result is that relatively few mathematicians refer to themselves as "geometers." As a consequence, when curricular changes are made, geometry's "interests" are rarely protected by persons steeped in either geometry or its methods.

Since most college faculty today were taught a rigorous theorem/proof geometry course in high school, and if they took geometry in college it was likely to have been axiomatically developed projective, Euclidean, or non-Euclidean geometry, it is easy for non-geometers to have the impression that axiomatics are central to current concerns of geometers. Nothing could be further from the truth (see Appendix A). Yet a great gulf exists between the work that geometers currently are doing and the perception, even among practicing mathematicians, about what is currently of interest to geometers. Most current researchers in geometry have little interest in axiomatics but are, rather, interested in developing new questions and methodologies, which, while nominally in the domain of Euclidean geometry, are very far from the traditional concern with the metric properties of circles, triangles, and quadrilaterals.

While Morley's Theorem that the points where the trisector lines of the angles of an arbitrary triangle meet are the vertices of an equilateral triangle (see Figure 1) is a dramatic example of 19th-century geometry (though it belongs to 20th-century geometry!), Chvatal's Theorem that $\lfloor n/3 \rfloor$ point surveillance devices are sometimes necessary and always sufficient to see the boundary of an n-sided polygon (see Figure 2) is a dramatic example of recent elementary geometry. Although Morley's Theorem has many generalizations, none have true research significance for new ideas or methods. By contrast, Chvatal's Theorem inspired a book-length treatment of related problems [Joseph O'Rourke, *Art Gallery Theorems and Algorithms* (Oxford University Press, 1987)], and leads in many directions of interest for mathematics and external applications. These problems are simple enough to explain to middle or high school students and yet generate many new results, methods, and potential

application. Fortunately, in addition to the book noted above, other important elementary expositions of recent accessible work in geometry are becoming available.
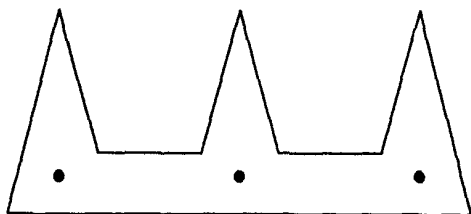
CHVATAL'S THEOREM: $\lfloor \frac{n}{3} \rfloor$ guards are sometimes necessary to "see" the interior of an n-gon.

Figure 1
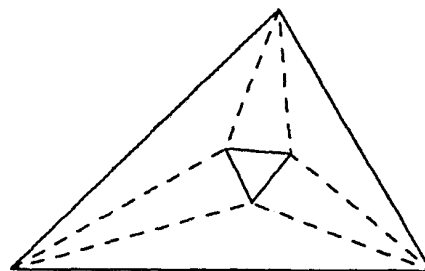
MORLEY'S THEOREM: The angle trisectors of a triangle determine an equilateral triangle.

Figure 2

Another possible avenue for generating excitement about geometry lies in the rapidly growing field of "visualization." The advent of supercomputers has made possible "computation intensive" solutions to geometric questions. More specifically, it is now possible to draw images of complicated dynamical systems. A good example of the interplay of computers and geometry at this level can be found in the book *Computers and the Imagination* by C. Pickover.

Question 6: Should colleges and universities develop an applied-oriented masters degree in the area of geometry? If such degrees were developed, what consequences would there be for the teaching of undergraduate geometry courses?

Background: In August 1989, the Society for Industrial and Applied Mathematics sponsored a conference on Applied Geometry. This conference, held in Albany, NY, was intriguing for the broad nature of the geometric methods and geometric problems that were raised. Some examples include: problems in solid modelling for designing better machine tools, automobiles, and planes; problems in robotics such as local and global motion planning; problems in computer vision such as object recognition; problems in computer graphics such as hidden line removal; problems in image processing such as how to remove blurs from an image; problems in geometric algorithms; problems in computational geometry. Yet what was noteworthy was that relatively few conferees taught in college or university mathematics departments. Most of the people who attended the conference would have described themselves as engineers or computer scientists and, if they were mathematicians, they were working in private industry rather than in university mathematics departments. Although many new methods and ideas were discussed at this conference, many classical methods were alluded to: projective geometry, geometric transformations, barycentric coordinates, splines, polyhedra, Voronoi cells, etc. In some cases it sounded as if the wheel was being re-invented for lack of knowledge of what geometers had accomplished in the past.

Professor Louis Billera (Cornell University) speculated during the conference about whether the circle of ideas that emerged as so useful in these seemingly disparate (but mathematically linked) areas could profit from practitioners with specialized masters degree training in "applied geometry." This question raises many interesting issues.

Historically, careers for undergraduate mathematics majors included high school teaching, actuarial positions, work in industry or government laboratories, work related to computing, operations research positions, college teaching, statistical environments (e.g., Census Department, Labor Department), etc. In order to obtain positions for many of these professions, additional graduate training in mathematics would be necessary, although not necessarily a master's degree in mathematics. For example, a student might go on to get a master's degree in education, mathematics education, statistics, operations research, computer science, decision science, or applied mathematics. The problem, as perceived by many undergraduates, boils down to the fact that there is no section of the want ads of major newspapers which lists ads for mathematicians. In contrast there are sections of want ads for such related fields as engineering and computer specialists.

Many mathematics majors are unclear concerning the value of a mathematics major as a career preparation (especially if they are not interested in becoming teachers at any level). They are also unclear as to what careers a (non-specialized) Master's Degree in Mathematics will enable them to pursue that could not be pursued on the basis of an undergraduate degree alone. If "applied geometry" was a specialty that had clearly identifiable associated career opportunities, the availability of master's degrees in this area would be an additional reason for mathematically inclined students to major in mathematics rather than allied computer, engineering, or scientific fields. Not only might the availability of such degree programs foster greater interest in undergraduate geometry courses, but it also might help forge links between undergraduate and graduate institutions.

Recent Attempts at Geometry Reform

Periodically, geometry makes a great leap forward at the research level, and yet these dramatic changes of direction in research seem to have little effect on the pedagogy of geometry. For example, Joan Richards in her book *Mathematical Visions: The Pursuit of Geometry in Victorian England*, documents the tortuous path by which concepts emerging from discoveries of Lobachevsky, Bolyai, and Klein entered schools at different levels and the mathematics community itself. There are many parallels with the current situation. There have been attempts in the past to lessen the influence of deductive Euclidean and non-Euclidean geometry on the curriculum. Often these reforms have foundered on what is perhaps geometry's greatest strength: its ability to illuminate such a broad collection of phenomena. A typical concern (voiced relatively recently by Edwin Moise) is that at least Euclid is a "theory." If we cut (deductive) Euclid from the curriculum, we will be left with a potpourri of loosely related topics.

In 1967, a conference of geometers was held in Santa Barbara under the auspices of CUPM to discuss future directions for the geometry curriculum. However, it is difficult to assess exactly what effects on curriculum resulted from the conference. More recently, the Consortium for Mathematics and Its Applications (COMAP) held a conference in Boston to study the possibility of changes in the undergraduate survey course and related phenomena. The recommendations from this conference appear in Appendix A. The proceedings of this conference, entitled *Geometry's Future*, also contains an enumeration of various sub-topics in geometry and a brief bibliography for these subtopics. (See J. Malkevitch, "Geometry in Utopia.")

The substantial excerpts from the electronic mail discussions reproduced below reveal a variety of creative ideas to deal with curriculum reform for geometry. However, they also show widespread disagreement about what form these changes should take and how to go about making these changes. We hope that this dialogue will be just the first step towards effective and creative reform.

Dear Geometers:

The quotations that follow are lightly edited excerpts from the electronic discussion under-
taken by this Focus Group. They provide various perspectives on the questions raised in the
introduction.

I would like to begin with questions about an applied masters degree. I first heard this notion from Lou Billera, though my conception of it and the reason why it appeals as an idea may be very different from his. As we all know you do not find want ads in the *New York Times* under the listing mathematician. Many students are confused as to degree options available. In particular, what jobs are there if one does not get a Ph.D.? At my school, where most of the students are minority, math majors go on to do actuarial work, high school teaching, banking, but rarely go for a Ph.D. Most of our mathematics majors are students who wanted to become engineers but were unable to do well in physics. These students would benefit, as would others who prefer mathematics to engineering, if they could be prepared by a mathematics program for a job. Applied masters degrees in geometry which would combine undergraduate and masters training in computing and mathematics in such areas as computer graphics, robotics, solid modelling, etc., seems to me a way of encouraging students perhaps currently lost to mathematics to stay with mathematics, and study lots of geometry too.

—Joe Malkevitch, York College, CUNY

The first wave of thinking about the role of mathematics in computer science always seems to yield "discrete mathematics." But in three important areas of computer science—vision, robotics, and graphics—the answer is "continuous mathematics," especially geometry and geometrically oriented linear algebra and multivariable calculus. I have seen this first-hand in industrial research I have done in robotics and I see it in trying to teach a graphics course to undergraduates at Adelphi.

One consequence is that some computer science programs might like a geometry course for their majors. If there is no room for that, they might appreciate a more geometric flavor for the courses in linear algebra and calculus. (How many linear algebra courses deal with isometries, affine maps, and projective transformations?)

In regard to graduate work, one might suppose that a degree in applied geometry is too narrow to attract students. As an example that this may not be true, I mention that New York Polytechnic offers a masters degree in image processing (that may not be the exact title). Geometry is at least as broad as that. However, I think one needs to take into account that most people go to graduate school as a way of targeting a particular career opportunity. There is a "computer industry" but not a "mathematics industry" in quite the

same sense. Perhaps one might devise a degree which had a substantial amount of "hands on" computing, and get some cooperation from computer science faculty. Then if one found a good title and effective marketing (how to do that?) one might make it fly. However, it might be something more risky or ambitious than any individual school would want to try. Is there an NSF program to subsidize ventures like this?

—Walter Meyer, Adelphi University

I have some additional thoughts to share about geometry and its position in the mathematics majors. Different mathematics majors, depending on their career goals have different needs. Historically, it seems to me that overly great concern has been given to undergraduate training in relation to future Ph.D.'s in mathematics. Most undergraduate majors do not go on to get Ph.D.'s. We are at a point where very few students show an interest in studying mathematics (no less geometry) in college. I believe not only can we be doing more to encourage students to major in mathematics, but also to study more mathematics as part of their major.

The first course a student sees, typically, is calculus. This course has had interesting geometry all but squeezed out in recent years. (There was a time when analytic geometry preceded calculus and many geometers enjoyed that, but I do not think we can turn back that clock, nor should we.) However, my point is this: If we are thinking about success for a Ph.D. program and a student appears lack-luster in calculus, many teachers will encourage students to try a different profession. Yet geometry skill and devotion are often exhibited independent of performance and skill in many of the mainstream undergraduate mathematics courses.

We need a way to identify talent and interest in geometry. One of geometry's appeals is that many parts require relatively little prerequisites. The course that Thurston, Doyle, and Conway are teaching at Princeton, open both to mathematics and non-mathematics majors, is intriguing in this regard since it is a serious experience for mathematical types, allows an early experience with geometry, and offers the chance of attracting students who by their exposure to mainstream mathematics may have not even had a chance to see what mathematics, no less what geometry has to offer them.

If we are to include recent geometry in the undergraduate mathematics major, a significant change will have to be made in the way the survey course is taught. In courses in differential geometry, convexity, and other courses with specialized content, I am sure recent ideas are treated. (I question, however, if these courses include as much applications-oriented material as they might.) However, the survey course, which is the course students are likely to see if they see any geometry at all, rarely includes significant recent material. As intellectually exciting as I find the axiomatics tale, I think we should not be teaching this any more. Rather, we should allow students to get this material on the side in much the way that now we expect them to learn the complement of axiomatics on the side. Recent pure and applied developments in all the far corners of our subject suggest the wisdom of doing this.

—Joe Malkevitch

I totally agree with Joe's statement that the axiomatic approach to Euclidean geometry

is *not* the right way to go in a college geometry course, be it survey or otherwise. And I agree that more material on applications should be included. However, there is one tricky question, so far as the preparation of prospective high school teachers of mathematics is concerned. *If* they will still be expected to teach axiomatic plane geometry in high school, and *if* the only college geometry course that is taken by many of them does not cover this approach, then where are they to learn it?

—Vic Klee, University of Washington

The nature of geometry implies the nature of the geometry curriculum. From the material that has come in so far, it is clear that there is an enormous new breadth to geometry. I mean by that statement that there are parts to geometry that those of us who were educated twenty years ago never dreamed of. This is especially true of applications. Who would have dreamed that projective geometry would be applied in computer graphics, that differential geometry would have had so much success in control theory (or physics—but that was partially anticipated), or that there would be a subject called "computational geometry." I agree with Joe that "as intellectually exciting as I find the axiomatics tale, I think we should not be teaching ..." We need to construct a wider platform for geometry and geometric methods. The nature of geometry implies the nature of a geometry curriculum.

—Richard Millman, California State University

Joe Malkevitch wrote about the narrowness of calculus as an entry into modern mathematics. Also, as is generally true of teachers, "I am teaching you to be like me. This is the extent of my concept of a mathematician. So get a Ph.D. in mathematics, and specialize in hemi-demi-flipoids like I did."

Besides geometry, there are lots of subjects one could start out in and get a different view of the subject. Well-talked-over is discrete mathematics versus continuous mathematics. This has become a big issue because of computer science. Calculus as a first course in college-level mathematics is like Latin, Greek, and Hebrew in the classical education of yesteryear. You had to study these subjects and do well in them before you were deemed good enough to study anything else. Today, pre-meds have to take calculus before they are allowed to study pill-pushing. I think a course in discrete mathematics might be a better choice.

There is a very great need to formulate geometry courses for computer users. For example, computers are used in architecture to make pictures of buildings (and their shadows) which can be twisted and turned to different vantage points as they are exposed to sources of light with various intensities. You can see what a building looks like as you walk past it. Given a mathematical description of an object, how do you write the program which gives the view from one vantage point or another, and how does light from a given source change the picture? There is a lot of geometry in this.

Also, archeology courses (in Philadelphia) involve building sites brick by brick using the computer. There is a lot of geometry in this too. There are problems in oceanography where soundings have been made at certain points on the surface of a body of water, so the depth is known at this spot and at other spots too. Try to describe the surface of the ground

under the water fitting these measurements. More geometry. Given a bunch of points in the plane, calculate the center and radius of the smallest circle that encloses all the points. How about the smallest pentagon? Calculate the convex hull of a given set of points. Classical problems in computational geometry. Image processing. Pictures of things. Get a picture of a regular icosahedron on the screen of a Macintosh. It might take somebody who knows lots of geometry quite a while to do this.

There is a need for a new course covering this type of geometry. The mother-of-all discrete mathematics books was written by Kemeny, Snell, and Thompson. After this great classic, a course was defined that spread to every college and university. The time is right for a book like this in geometry. I wonder what should be in the table of contents. Perspective drawing. Projections. Co-ordinate geometry. Hidden line algorithms. Other topics from computational geometry. Applications to various things—to medicine, architecture, archeology, making movies, art, you name it.

I think geometry fell from favor because of axiomatics. Because geometry is so close to intuition, we often feel we understand more easily without any formal model. One can solve geometric problems without thinking of the axioms. It is another game to prove things. Here the axioms play a role. To get a nice picture of the regular pentagonal dodecahedron on the computer screen which can be turned every which way with 'hidden lines' dotted as the solid turns is quite a project—but no axioms are needed.

—David Klarner, University of Nebraska

As I see it, we are trying to address the educational needs of at least three constituencies: the future teacher, the future mathematician, and the job seeker. At the same time there is discussion of "the course." Is it really possible to design a course that would be appropriate for all three groups? Or would it be more productive to design and press for the adoption of more geometry courses?

Another puzzle is that of the chicken and the egg. True, most teachers will be expected to teach axiomatics. But this may not be what they ought to be teaching—even in high school. I think the colleges and universities have to take the lead here. The Admissions Office here at Smith is frequently told that high schools follow the college's lead: if we require four years of mathematics, the schools will offer it; if we don't, the schools might bend to other pressures. The content of the geometry course is a more complicated issue, because the high schools feel they owe it to the students and their parents to prepare them for the SAT's, which seem to be rather hard to modify. But then if there is any hope of modifying them, it will have to be due to leadership from us. To summarize: the existing high school geometry curriculum should not deter us from teaching more appropriate geometry to prospective teachers.

—Marjorie Senechal, Smith College

I have lived in the Netherlands a total of three years and saw there a completely different educational setup. For one thing, there are about 13 million Dutch people, so the country is like a moderate sized state here. But the administration of the universities is a federal thing in the Netherlands. A percentage of GNP is used to fund them. Further, the universities are a federal resource, so they are used in administering the schools at all levels. So picking

textbooks for the schools is done by people who know something about the subjects. Lots of top mathematicians get questioned about education at the lower levels. Since things are organized this way, people in the universities already understand their relationship to education, and they know what they are supposed to do about it.

Compared to the Netherlands, we suffer from a lack of definition of what we should do, and how to go about doing it. This lack of definition hurts when we try to settle what should be done about geometry. I still have my high school geometry book, and I must say the axiomatic presentation is awfully confused. Probably Peano's axioms for the natural numbers are easier to understand and use than the geometric axioms. In fact, Hilbert was the first to give a proper formulation of classical geometry. As far as reasoning in a rule-based system is concerned, there are lots of things you can do with chess problems. The disadvantage of axioms in geometry is that we are convinced that our intuitive notions are already adequate.

Well, with all this discussion, what do we the university experts think should be done? Is geometry the place to introduce axioms? As far as the history of ideas is concerned, that is where it started, and people ought to know about it. But there are simpler systems than Euclidean Geometry that might be used to illustrate the axiomatic approach. As far as problem solving is concerned, geometric or otherwise, I don't have much use for axioms.

Geometrize!

—David Klarner

The point that geometry is useful in new and unexpected ways has already been made well by Walter Meyer and David Klarner. We could describe most of this as "applications to computer science," but it really is applied geometry that happens to have been developed by computer scientists (for the most part) because they need it and it's not coming from mathematics. The question is, I think, how do we get on the bandwagon before it goes much further without us?

If you are interested in learning about some of what's going on in applied geometry, attend one of the upcoming SIAM meetings on "Computer-Aided Geometric Design" (CAGD). I went to one about four years ago. It was a real inspiration to see how vital a subject geometry had become, although you wouldn't know it if you looked at the mathematics curriculum at most universities. In addition to the CAGD people, there were talks on robotics, computational geometry, computational algebraic geometry, and automatic geometry theorem proving (Groebner basis theory), and polyhedral combinatorial optimization.

Two things about that meeting struck me particularly. The first was that the more interesting theoretical work being discussed had not been done in universities but in industrial research labs, General Electric and General Motors in particular, by people who seemed to have a strong background in various areas of geometry. Strangely enough, at least some few industrial research groups are able to take a longer-term view in the development of geometric methods. University groups that reported (for example in robotics or computer graphics) felt the need to have computer code or pretty pictures to show for their efforts.

The second aspect of the meeting that influenced my thinking came when Branko Grünbaum, at the end of his talk, pointed out to the assembled group that the pipeline for trained (and educated!) people to work in these exciting new areas was essentially

empty. That nowhere in this country were there undergraduate students getting the necessary geometric background to be able to participate in these efforts. (Branko, please correct me if I've misrepresented what you said.) Since then I've thought about, talked about, but essentially done nothing about developing an upper-class course in applied geometry.

What I have in mind is something that already exists in many places in applied algebra. Several years ago I had a hand in introducing such an algebra course here. Since I taught it the first time, it has been given each year, now each semester, as well as during the summers (occasionally in multiple sections). It provides an introduction to the basic stuff of modern algebra (groups, rings, and fields) but does so in the context of more-or-less live applications: symmetry and Pólya enumeration, cryptography, error-correcting codes, etc. In addition to being popular with the students (some of whom are induced to take the "real" modern algebra course as a follow-up) it has been a real hit with the pure algebraists, who for a change like the idea of teaching something whose relevance is clear to the students. (It is so much of a hit that I'll only get to teach it myself for the second time next year.)

Applied geometry is a harder course to implement. There are no books (anyone interested?) and it's not clear what should be in them. As opposed to modern algebra, what is currently being taught in geometry in most places is irrelevant to these applications. The problem here is to assemble a list of geometric topics together with related live applications through which they can be introduced. Here the emphasis is on "live:" most students will not see the relevance of perspective drawing, Escher prints, and theoretical physics (as interesting as all of this is) to their ability to earn a paycheck. As crass as this may sound, it is exactly such considerations that leads most students to choose computer science over mathematics as a major.

The final issue I want to discuss was raised by Vic Klee and Marjorie Senechal, namely how to fit new ideas into the existing curriculum. I don't have any complete and coherent list of topics for a one-semester applied geometry course, but I think it's worth trying to make one. More likely, such a course will try to be at least a year long. So how to fit this into a mathematics major that now barely has any room even for axiomatic geometry (a logic course in disguise)? I don't propose we wait until the SAT's are reformed; I don't think we have the luxury of waiting that long.

Why don't we just require more mathematics of mathematics majors? It's my experience that mathematics has been one of the least demanding of majors in terms of requirements (compare with engineering!). This explains why in some places the mathematics major is popular with pre-meds, who need to take a lot of varied courses. When I was an undergraduate, I had enough free time to get the equivalent of a major in psychology on the side. Currently, I have an undergraduate advisee (who will be in Seattle next year as a mathematics graduate student) who managed to get a triple major (mathematics, operations research, and German). Surely having mathematics majors get a minor in computer science or operations research is a good thing, but why not offer the chance to minor in applied geometry (maybe some computer science and operations research students will join them). If you have ever done graduate admissions you know the difference in preparation of the average U.S. undergraduate mathematics major and that of mathematics students from virtually anywhere else.

—Louis Billera, Cornell University

Vic Klee raises a critical issue. If we change the survey course, whose historical audience has included a significant number of future high school teachers, how will such teacher trainees cope with what they are expected to teach currently in high school? This same point has troubled me when I have taught the survey course on and off for the last twenty years. My solution was to teach a lot of geometry that I felt was no longer the best choice for the students. Now, however, I think I see a solution to this problem, and it grows out of some of the teaching reform ideas, including writing across the curriculum. The idea is to have students in the survey course read on their own, not the details and the mathematics of the wonderful foundational story, but its history and an outline of the details.

There are a few books that deal with development of the interplay between the mathematics of the conception of space and the physicists' view. Two such are *Ideas of Space* by Jeremy Gray (Oxford University Press, 1980), and *Space Through the Ages* by Cornelius Lanczos (Academic Press, 1969). Both these books try to chart, in a conversational but not unmathematical way, the historical, mathematical, philosophical, and physical aspects of the interplay between geometry and space (in the physical sense). Another is H. Wolfe's book on *Non-Euclidean and Euclidean Geometry* (which is out of print, but which starts with a long introduction about these matters). I would ask the students to read one such book or collection of materials and write a "report" about it. This seems to kill two birds with one stone. It gives students an opportunity to see this material but more from a cultural perspective than a mathematical one, and it gives them an opportunity to read and write about mathematics in an historical framework.

I learned this type of thing in high school and in some college courses, whereas I learned the exciting ideas about tilings, Reuleaux triangles, curvature, and minimal surfaces, etc., etc., etc., from a math club. Why not reverse this? Put that other material at the fringes and get our eyes back on a kind of geometry that makes for a more meaningful current experience.

—Joe Malkevitch

I have taught a course on computational geometry four times now, twice to graduate students and undergraduates at Johns Hopkins, and twice to undergraduates only at Smith College. The course was offered in computer science departments, but some mathematics majors enrolled. Although what I cover may include more algorithms than most of you might prefer in an applied geometry course, I think it demonstrates the possibility of teaching such a course.

My experience is that the students can get *very* excited on these topics. The problems are motivated by applications, the relationships can be visualized, use of the computer makes it all concrete, and the easily-grasped open problems tantalize.

I cannot claim to be filling the empty pipeline that Lou Billera cites Branko Grünbaum as identifying, but I believe a trickle is being produced by those teaching computational geometry, mainly from within computer science departments. Several former students in these classes have maintained their interest in geometry for nearly a decade now.

If you will pardon the self-advertisement, I am currently writing a textbook for an undergraduate course in computational geometry. The prerequisites are basic programming, calculus, and a smidgen of linear algebra. Here are the five main topic headings, with a few

sub-topics identified:

1. Partitioning Polygons. *Includes* art gallery theorems, polygon triangulation.
2. Convex Hulls. *Includes* 3-polytopes, Euler's relation.
3. Voronoi Diagrams. *Includes* Delaunay triangulations as projections of convex hulls, minimum spanning trees.
4. Arrangements of Lines and Planes. *Includes* connection to Voronoi diagrams.
5. Robotics. *Includes* shortest path algorithms, Minkowski sum, Collins' cell decomposition.

—Joe O'Rourke, Smith College

I have been reading people's comments with great interest. It does seem that part of the problem in thinking about geometry courses is that not only is there a diversity of constituencies but also there is quite a range of subject matter that suggests itself for undergraduate geometry.

In order not to get lost in all these possibilities, for the moment I would like to comment on a single case: students taking freshman and sophomore "paycheck" mathematics: calculus, linear algebra, and differential equations but not much beyond. These students are important, not only because they form the bulk of our mathematical customers in the undergraduate mathematics business, but also because they will go on to form the main body of scientifically literate citizenry.

While I would love to introduce a course called Geometry or Applied Geometry to be taken by all these students, I don't think we should ignore the possibility of putting back analytic geometry (or a modern substitute) into the standard introductory mathematics courses. This may not be a substitute for real geometry courses, but it can be a valuable adjunct and an improvement over the *status quo*.

I would like to report on one experiment that has gone on here at the University of Washington this year and then describe what I plan to do next year.

The third quarter of our freshman calculus course has been purged of other topics and is now devoted exclusively to multivariable calculus (differential calculus and multiple integrals but no div, curl, or Stokes theorem). The extra time is now devoted to a more extensive treatment of elementary vector geometry than was possible before; there is time to give some intermediate problems about lines and planes instead of just checking on the plugging-in of definitions. This course is taken by all the students on the main calculus track.

This year, as a companion to this third-quarter calculus course, two of my colleagues (Caspar Curjel and John Sylvester) have been running a one-credit computer lab course which concentrates on visualization in multivariable calculus. The students do not do what one normally thinks of as calculus problems. Instead they rotate pictures of 3-D objects (lines and planes and surfaces) with the mouse and try to analyze them, finding coordinates of points, directions of line segments, intersections, tangent planes, by looking rather than by using formulas. It is surprising how challenging this is.

For my own part, I have proposed to teach a similar lab next year to be attached to our linear algebra course, which suffers from a dearth of applications. My intention is to teach elementary linear algebra with a focus on geometric applications, looking at practical

problems arising in engineering or computer science that can be solved using rotations, projections, transformations, etc. At the moment this is an experiment. In the future such a course in Linear Algebra with Geometric Applications could at least be an alternate to the regular linear algebra course if not a substitute for it.

To sum up, while I think we should be looking for ways to install geometry courses in the undergraduate curriculum, I don't think we should ignore the possibility of returning some of the geometric content to the standard calculus and linear algebra courses as well.

—Jim King, University of Washington

Although not everyone has been heard from yet, there seems so far to be little debate about the intellectual thrust of what a budding geometry movement should be: non-axiomatic, hands-on (including computers), with a strong flavor of applications and algorithms. The troublesome issues (so far) deal with strategies:

1. If we change geometry drastically at the college level, what happens to students who will have to teach the old stuff as high school teachers?
2. Should we put more geometry into the "paycheck" courses (great phrase!). Although the question was not raised in King's message, one might wonder whether that would be an alternative to a revitalized full course in geometry, or take the steam out of attempts to create one.

This leads me to the thought that we ought not to worry too much that we might upset some apple carts if we push forward in the wrong place. The curricular system is highly resistant to change. Anyone who has attended geometry sessions of mathematics meetings recently knows that there are many people teaching this subject who seem quite happy with very traditional ideas. We should push forward at both the high school and college levels; with separate courses or more geometry in old courses; with "applied geometry" and "computational geometry"—secure in the knowledge that the last thing that we are likely to do is to create a rapid revolution.

—Walter Meyer

The author introduces himself: First of all, I should confess that I am not a card-carrying geometer, but merely a defrocked algebraist who loves geometry. Moreover, I've been spending the last several years on a project which has been producing educational materials for school geometry (computer programs, computer-generated videotapes of three-dimensional figures, workbooks). Doris Schattschneider has been the real geometer on the project, but she lacks reliable bitnet to defend herself.

The conclusion thus far should be that you can safely ignore the mildly heretical views which I will soon begin expressing. Fair enough, but you should also educate me, since I frequently rush in where true geometers fear to tread. For example, the editors of the new MAA publication on *Visualization in Mathematics* conned me into doing an article on geometry, since they were having trouble finding the real McCoy. Moreover, the education project has been reasonably successful and Doris can't do everything, so I end up talking with teachers and administrators about school geometry. So argue especially hard when you disagree with me.

The author confesses to certain unhealthy tendencies: Brace yourselves, for I am going to say a kind word about axiomatics. The last few times I've taught geometry, I based the course on Greenberg's *Euclidean and Non-Euclidean Geometries.* I like very much examining some false proofs of the parallel postulate, and otherwise exhibiting the troubles with Euclid. It seems to me that one can well-prepare students to deal with the major repair job done by Hilbert and that it's very good training to get students to do some work with that axiom system. (It is much better training than in lots of algebra, for example, since geometry words carry such strong connotations.) Where better to show what axiomatics is really good for— a method for expressing mathematics well-suited to examination for gaps in reasoning and inconsistencies. Of course, one should feel honor-bound to make clear to students that this is how mathematics is frequently conveyed, but not how it's created.

On the other hand, only a sado-masochist would try to drag students through a rigid development from Hilbert's axioms, so I skip around when they start to turn purple. Greenberg is very good in not bringing in non-Euclidean models for a long time, and student minds are visibly stretched as they try to argue about hyperbolic properties from an axiomatic framework.

In sum, I think that if students are prepared to reason carefully within an axiom system, and if they see the necessity for doing so, it can be a very good thing. On the other hand, I readily admit that one can have too much of a good thing.

The biggest disadvantage to my (semi-)axiomatic approach is that it takes time, so I can't cover all the wonderful geometry I'd like to. (I've found this to be a problem with all the other approaches I've tried, too.) A final advantage of Greenberg's approach is that he very beautifully conveys the history of the discovery of non-Euclidean geometries, surely one of the most wonderful stories in all of mathematics. It's just grand to get in such splendid mathematics along with some history and philosophy. (Of course, I do take lots of side trips and detours to view some other geometry vistas.) I need to be convinced that I can do better for my general clientele than this approach, even though it is based on axiomatics.

—Eugene Klotz, Swarthmore College

I think our discussions so far fall into two categories: global issues and local issues. At the global level we have to consider how the nature of high school education has changed in the last thirty years (e.g., concern with larger percentage of successful graduates) and how college education has changed during the same period (e.g., larger volume of students, more concern on the part of students with how they will earn a living and how much they will earn, and the fact that computer science is a credible alternative for people with mathematical inclinations). Along with this there is the issue of what kinds of people study mathematics— their interests and motivations. I too have noticed the tendency for mathematicians to see the world in their own image. "Since I got turned on to mathematics by seeing Euclid in the tenth grade, it must be a good thing for everyone." Tradition is a very strong force at this level of analysis.

Local concerns are those dealing with how to restore geometry to a more central place in linear algebra and calculus; how to meet the needs of computer scientists and others with geometry courses taught within mathematics departments; how to treat the needs of future high school teachers; how to take advantage of the new technologies to teach geometry.

Several people have raised the idea of a new type of applied-oriented course, and tossed out possible topics, but were nervous about the great variety of possibilities. How about several independent attempts at a curriculum for such a course to see if there is a common core of topics people see as being significant?

Gene raises important issues in his comments on the fun, success, and value of doing axiomatics. Gene is too modest about what he has contributed to geometry. Ironically, as geometry has advanced and geometry courses have become traditional, there has been a growing gap between what geometers find interesting and what is being taught. Nevertheless, despite whatever value I used to see in teaching axiomatics for high school teachers or for mathematics majors with their eye on the Ph.D., I now believe that I can achieve my old goals and more with the kind of course I outline in "Geometry: Yesterday, Today, and Tomorrow."

—Joe Malkevitch

Having defended something which everyone else in the conference apparently dislikes, I'll now compound matters by attacking a consensus favorite. To salvage a wee bit of credibility here, let me relate some history. A long time ago I got the bug to teach applied algebra. My attempts were less than successful (I envy Louis Billera and his success). I had to cut back on the pure algebra which I normally covered, and in addition either I had to water down an example to the point where it appeared contrived to the students, or I had to load on so much detail from another subject that it only appealed to a small minority of the students. Since we already had developed enough traditional ways of turning off students, I abandoned this approach and went back to teaching "Groups, Rings, 'n Things," which was at least attractive to some very desirable students.

It seems clear to me that there should be real possibilities for courses with titles like Computer Science Geometry, but I question the viability of courses entitled Applied Geometry for Everyone. Beware the siren song of "relevance" which all too frequently attracts faculty but not students.

Last week our department had a discussion with students as to how we might improve our courses and make them more accessible. One faculty member suggested more applications. A student with impeccable taste voiced her concern that applications might be given in such a fashion as to convey the feeling that the mathematics wasn't interesting enough in itself. I'm on her side. I think that the contents of a mathematics course should be taken from the intersection of what students need to know, what is good for students, and wonderful mathematics. In geometry, I've always found the last subset so large that one can accommodate all sorts of changes to the first two.

From what I've seen, there's some very interesting new applied geometry. For example, I agree wholeheartedly with the favorable comments already made about computational geometry. The problem, as I see it, is that one has to be very careful—mediocre applied mathematics could force out some beautiful pure mathematics. This would be particularly unfortunate at a time when new possibilities are emerging for making a lot of classical geometry more accessible. But that will be the subject of another harangue.

—Eugene Klotz

I do not particularly wish to become embroiled in the pure mathematics—applied mathematics wars. From my point of view, such questions ultimately come down to matters of taste. What someone may refer to as "mediocre" applied mathematics may well seem aesthetically limited to me yet still be both non-trivial and important. After all, mathematics is used in the real world to the advantage of us all. If one decides to teach a geometry course and not include applications at all, I see no problem with this. I am a big believer in courses being most effective when the instructor is passionate about what he or she is teaching.

However, I see mathematics as having a flow and sense of direction as it grows. Some parts of what we as geometers know already is more or less important to the discovery and understanding of what we hope to know tomorrow. Sure there are new theorems about quadrilaterals to be discovered. But is this the best preparation for a student with an interest in geometry who is Ph.D.-bound? If the student is not Ph.D.-bound, but wishes to study actuarial science or work for an engineering firm, then there are criteria that could be applied to select optimal content for students at different levels.

—Joe Malkevitch

Last week I attended the conference in Orlando in honor of the 80th birthday of Howard Eves. There were over 150 participants from all over the country, ranging from middle school teachers to university professors and retired mathematicians. The talks represented Eves' fields of interest—geometry, pedagogy, history, and problems (as in problem columns in journals). It was a very pleasant and a very interesting meeting, with a lively, concerned audience, but I was struck by what seemed to me to be a lack of awareness of the scope of geometry and its burgeoning future.

I told them that mathematicians build gardens whose walls are axioms and definitions and propositions, and then busily cultivate the gardens, but often a plant escapes outside the walls and blooms more vigorously outside, or changes its characteristics, and we need to be aware of this. But nobody else was talking this way. On the other hand, it is clear that they all love geometry as they perceive it and are concerned about protecting its place in the curriculum. It seems to me that these folks—geometers all—need lots of accessible expository articles on contemporary geometry, and of course good texts. If this is correct, then it follows that we should be writing articles for journals like the *Monthly* and *Mathematics Magazine* and *Quant* and all the others, as well as writing interesting textbooks on a variety of geometrical topics. And the MAA should publish something like "Studies in Contemporary Geometry" aimed at the college teacher level (including those in community colleges). Maybe it already is doing something like this, but if it isn't, then it should.

Another useful thing—although a lot of work for somebody—would be a "Geometry Newsletter" with reports of conferences, lists of new books, discussions of models and visualization and other things, and a letters column.

It seems to me that we needn't worry much about axioms vs. applications, this vs. that. The world of geometry is enormous and we need *all* of it. The vs. problem only matters if you are trying to write *the* text for *the* course. But maybe in the future there will be many courses.

While working on *On the Shoulders of Giants* I was concerned about whether and how this material would reach the teachers. I don't know how it happened, but it did. Reports

reaching me say they are reading it and talking about it. Maybe one outcome of this e-mail conference could be something like that—a collection of "strands" of geometrical thought. This would be a good theme for the MAA volume I mentioned a few paragraphs ago.

—*Marjorie Senechal*

The view from the mathematics education trenches is not all bleak when it comes to geometry. I would even expect that some of the favorable aspects might have a "trickle up" effect for college teaching. For example, there are some new and less-moribund texts starting to appear, most notably Mike Serra's *Discovering Geometry.* Moreover, some new computer programs are arriving which should have serious impact.

The first generation of these programs were those in Judah Schwartz's "Geometric Supposer" series. Although one can disagree with their procrustian Apple II design, they have nonetheless inspired a number of high school teachers and have an admirable user network.

It is the second-generation programs which I believe will be more revolutionary—programs such as the French "Cabri Geometrie" and our "Geometer's Sketchpad." (NB: Although I was involved in the production of the latter, the ideas involved are in the air—witness Cabri—and it is the type of program which should be of educational importance, not any particular one.)

What these programs can do is make Euclidean constructions easy, both for students and for teachers (some have unlimited undo and redo, so one can prepare a presentation and then easily go through it step-by-step). Beautiful "ancient" geometry which was always a pain to present at the elementary level (the 9-point circle, for example), can now stir student interest and not foment student revolt. These new programs not only facilitate the construction of figures that students find interesting, but they also allow the figures to be deformed so that students can search for the conditions under which a particular theorem holds. Some of these programs also show loci of points, envelopes of lines, and the like, which opens up lovely new vistas.

One effect of this will be the discovery of some new geometry—and the rediscovery of lots more—by mere high school kids. This is already happening with the Supposer, and the second-generation programs should create a deluge. Be prepared for endless questions as to the originality of elementary Euclidean results from high school teachers and students! (In particular, bet on a proliferation of loci problems, and Joe may even receive some new results about quadrilaterals—which he may not mind if they come from high school kids.)

One cloud I see on the horizon is that while the new programs encourage constructions and play, they don't seem to encourage proofs. For some students it seems to be enough to see that something holds; they still have to be convinced that it's important to ask "why?" I am busily looking for a construction which appears to hold on the computer screen, but which is in fact incorrect. (Thus far I am unsuccessful in geometry; for graphing programs, you can sometimes create a graph which looks just like $y = x$, but which is really a step function with many, many steps, but seen from a far distance. This can cause healthy mistrust in the infernal machines.)

The college geometry community would do well to see if they can provide direction as to what might constitute an appropriate revitalized high school geometry curriculum (if they can agree) since, if I am correct, more students may soon be attracted to geometry.

(They should also try some of the new software in classes. I've had a lot of fun with various hyperbolic models, inversion, projective generation of conics, transformations, etc.; so have my students. Both "The Geometer's Sketchpad" and "Cabri" are Macintosh programs.)

—*Eugene Klotz*

Gene Klotz asked for a construction which appears to hold on the computer screen, but which is in fact incorrect. Stan Wagon has a nice example of this in *Mathematica in Action*, p. 53. He plots the curve traced by the centroid of a Reuleaux triangle rotating inside a square, and it looks so like a circle that one might be convinced that it must be. But it is, in fact, composed of pieces of four congruent ellipses.

—*Joe O'Rourke*

Gene has raised the importance of computer environments for carrying out explorations of geometry. Not only are there nice tools such as the "Geometric Supposer" and the "Geometric Sketchpad," but there are also *Logo* and *Mathematica*. As Joe O'Rourke has pointed out, Stan Wagon has some lovely illustrations in his new book of how this can be done, and there is a large literature at the high school level, as Gene mentioned dealing with the "Supposer." I am a big fan of *Logo* and I feel it is unfortunate that more *Logo* is not done at the college level. There are other geometric toolkits in the areas of computational geometry and graph theory that are in the experimental stage of development. I feel that these tools can be employed to foster interest, exploration, and breadth.

—*Joe Malkevitch*

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Appendix A: COMAP Geometry Conference

In recent years, there has been a tremendous surge in research in geometry. This surge has been the consequence of the development of new methods, the refinement of old ones, and the stimulation of new ideas both from within mathematics and from other disciplines, including computer science. Yet during this period of growth, education in geometry has remained stagnant. Not only are few of the new ideas in geometry being taught, but also fewer students are studying geometry.

In March 1990, a group of college and university researchers and educators in geometry met to assess the directions of education and to make suggestions for invigorating it. These individuals represented a wide variety of branches of geometry as well as a wide spectrum of institutions. Discussions ensued on the causes of the decline in geometry education and on the steps that might be taken at all grade levels (K–graduate school) to energize the teaching of it. Special attention was given to the content of the survey course in geometry taught in many universities and colleges. This course has historically been taken by a large number of prospective high school teachers, and thus setting new directions for this course offers the hope of exposing future mathematics practitioners to new ideas in geometry, as well as for laying the basis for future changes in lower grades.

Despite the varied points of view expressed by the individuals who attended the conference, there was a broad core of common views, which, if implemented, can have a significant effect on geometry. This common core of views and recommendations is presented below.

These recommendations and the following article "Geometry: Yesterday, Today, and Tomorrow" by Joe Malkevitch are reproduced with permission from *Geometry's Future*, the proceedings of a March 1990 conference sponsored by COMAP, Inc. (57 Bedford Street, Suite 210, Lexington, MA 02173).