Conference Recommendations

Future directions for the teaching of geometry (especially for implementation in the college/university survey course):

- Geometric objects and concepts should be studied more from an experimental and inductive point of view rather than from an axiomatic point of view. (Results suggested by inductive approaches should be proved.)
- Combinatorial, topological, analytical, and computational aspects of geometry should be given equal footing with metric ideas.
- The broad applicability of geometry should be demonstrated: applications to business (linear programming and graph theory), to biology (knots and dynamical systems), to robotics (computational geometry and convexity), etc.
- A wide variety of computer environments should be explored (*Mathematica*, *Logo*, etc.) both as exploratory tools and for concept development.
- Recent developments in geometry should be included. (Geometry did not die with either Euclid or Bolyai and Lobachevsky.)
- The cross-fertilization of geometry with other parts of mathematics should be developed.
- The rich history of geometry and its practitioners should be shown. (Many of the greatest mathematicians of all time: Archimedes, Newton, Euler, Gauss, Poincaré, Hilbert, von Neumann, etc., have made significant contributions to geometry.)
- Both the depth and breadth of geometry should be treated. (Example: Knot theory, a part of geometry rarely discussed in either high school or survey geometry courses, connects with ideas in analysis, topology, algebra, etc., and is finding applications in biology and physics.)
- More use of diagrams and physical models as aids to conceptual development in geometry should be explored.
- Group learning methods, writing assignments, and projects should become an integral part of the format in which geometry is taught.
- More emphasis should be placed on central conceptual aspects of geometry, such as geometric transformations and their effects on point sets, distance concepts, surface concepts, etc.
- Mathematics departments should encourage prospective teachers to be exposed to both the depth and breadth of geometry.

Appendix B: Geometry: Yesterday, Today, and Tomorrow

by Joe Malkevitch, YORK COLLEGE, CUNY

Despite the increased pace of exciting developments in both the theory and applications of geometry in the last 40 years, it appears that less geometry is being taught in college today than was taught in the recent or distant past. The purpose of this paper is to examine this "paradox" and to study how the teaching of geometry in colleges affects what geometry is and can be taught in high school, grade school, and graduate school mathematics.

Geometry in Mathematics Departments Today

A perusal of recent college catalogues show mathematics departments listing (though not always regularly offering) a variety of geometric-based courses: Graph Theory, Differential Geometry, Convex Sets and Geometric Inequalities, Combinatorial Geometry, Projective Geometry, Topology, etc. In addition to courses such as these, many mathematics departments offer a survey course in geometry under a variety of titles. These include College Geometry, Euclidean and Non-Euclidean Geometry, Topics in Geometry, Modern Geometry, Geometric Structures, etc. It will be convenient to refer to the first type of course as a Geometry Course and the second type as a Survey Course. (Geometry also enters the curriculum in a variety of other courses including Calculus, Linear Algebra, Combinatorics, etc.) Although it is rare to require either of these types of courses of students majoring in mathematics, it is not uncommon for many mathematics departments to require a Survey Course or some Geometry Course from those mathematics majors planning to teach mathematics in secondary schools. This type of requirement reflects the fact that the "traditional high school curriculum" includes a year of study of geometry in the tenth grade. Thus, the geometry taught in college is closely tied through teacher preparation to the geometry taught in pre-college mathematics. To explain the decline in the teaching of geometry in college requires a digression.

Why Students Major in Mathematics in College

Most college mathematics majors fall into one of the following groups: students planning to enter graduate school to start in a program of doctoral studies in mathematics, students planning careers as high school (or sometimes intermediate school) teachers, students planning to pursue careers relating to computing, students planning actuarial careers, students planning to enter an "applied" master degree program (this is usually a terminal degree that does not result in the student pursuing the doctorate degree), and "others." Especially at colleges in large metropolitan areas, high school mathematics teachers have traditionally constituted a significant portion of the total number of mathematics majors. With the downturn in mathematics majors that was seen in many colleges during the period 1972-1988, one saw a dramatic reduction in the number of students preparing for careers as secondary school teachers. This reduction is ostensibly attributable to several phenomena. First, the dramatic decrease in the number of students in the school system during the period meant that many teachers currently in the profession were laid off. Second, the dramatic oversupply of mathematicians, engineers, etc., in the post-Apollo period made students wary of majoring in these subjects, and the high salaries paid in the computer science siphoned away many students with interests in mathematics. Third, the salaries of high school mathematics teachers relative to other professions that potential teachers of mathematics could enter became eroded.

When the downturn in mathematics enrollments in general and mathematics secondary school teachers in particular hit our colleges, the effect on the teaching of geometry courses was especially extreme. This is clearly related to the fact that the major group of students taking survey courses were future high school teachers. Even for geometry courses, loss of enrollment in high school teacher audiences resulted in decreased offerings. It is unfortunate that this diminished exposure to geometry for mathematics majors has come at a time of tremendous dynamism for geometry itself.

What is Geometry?

Before continuing with more detailed discussions, it may be useful to explain how the term geometry has been and will be used in this essay. In attempting to clarify what is meant by the term geometry, it is clear that the word "geometry" means different things to different audiences, including subgroups of the mathematics community itself.

To lay people, geometry is the study of the space and the shapes that they see in the world around them. Most lay people's exposure to geometry is the simple material on classification of shape that they learn about in grade school and the exposure to "pseudo-axiomatic" geometry in high school. Much of high school geometry is still highly concerned with the axiomatics and the proving of Euclidean theorems in a manner that has come to be described as two-column proofs. This refers to a series of statements and the reason for the statements in a second column. In recent years, there has been a growing movement toward a more "inductive" approach to geometry, spurred on in part by the development of such software packages as the "Geometric Supposer." However, this movement has been nearly exclusively concerned with the metric properties of triangles, quadrilaterals, and circles. Thus, to the non-mathematician, geometry has a very narrow meaning. Obviously, "geometry" has much richer connotations to members of the mathematics community.

However, even within the mathematics community, geometry means a surprisingly diverse number of things to different people. To some, geometry refers to those portions of mathematics (and mathematical physics) that deal with the mathematical structure of space, thereby involving a large variety of deep mathematical tools such as operator theory, partial differential equations, and Lie groups. To others, it refers to differential geometry and the topology of manifolds. Yet other groups think of it as meaning (though not exclusively) the emerging body of ideas dealing with discrete geometrical structures. As diverse as the meaning of the word geometry is, a remarkably large portion of the subject can be introduced and profitably pursued with a minimum amount of background and formal study of mathematics. In this sense, geometry differs greatly from other parts of modern mathematics such as functional analysis, ring theory, logic, algebraic topology, etc.

Here the word geometry will be used in its very broadest sense of all aspects of mathematics where visual information, diagrams, models, and understanding of space are involved or put to use. For an attempt to catalogue the breadth of ground entailed by this viewpoint, see Malkevitch [8]. It is noteworthy that a variety of rapidly emerging areas within mathematics and computer science have a major geometric component. In order to see how geometry fits in the college curriculum of the future, it will be useful to examine the traditional relationship between geometry and other parts of mathematics.

Geometry's Relation to Mathematics

It is interesting to note that although many areas of mathematics have first been developed in geometric form, these areas have often matured when they were algebratized. Examples include synthetic Euclidean geometry, projective geometry, block designs, catastrophe theory, etc. As important as geometry is both to geometers and mathematics, as a separate discipline, it has never been in the mainstream of mathematics, once mathematics as a subject for study was institutionalized in universities and colleges. In a pre-World War II university or college, during the period when the roots of the current renaissance in geometry were being laid out at the research level, there were fewer Geometry and Survey

Courses being taught than would have been the case from 1960-1975. Thus, a university during the 1920s or 1930s would have had courses in Analytic Geometry, Solid Analytic Geometry, Projective Geometry (perhaps in both synthetic and algebraic versions), and (old style) Differential Geometry. The wealth of geometry courses listed (though often untaught) at the college and university of today were uncommon then. In fact, at that time, no explicit survey course in geometry existed. (No equivalent of Howard Eve's pioneering *Survey of Geometry* (1963) with its curious forward- and backward-looking collection of topics existed before the War. The niche for high school teachers, trained then in "normal" schools or colleges, was filled by courses such as College Geometry or Modern Geometry. For a sample of the books of that era see Eves [1, p. 115]. Courses on convex sets, graph theory, groups and geometry, etc., virtually did not exist.)

Today, a standard introduction to mathematics for a graduate student pursuing a doctorate degree consists of a year of Real and Complex Analysis, a year of Abstract Algebra, and a year of Topology (with geometric aspects of the subject not necessarily emphasized). The teaching of topology often serves the role of hand-maiden for parts of Real and Complex Analysis. Judged by the dissertation titles that one sees listed in recent years by the American Mathematical Society, geometry is a relatively minor field at the fingers of most research. (Perhaps symptomatic of geometry's problems is that in the new 1990 mathematics subject classification list, the rapidly emerging area of computational geometry receives no listing.) The qualifier/preliminary examination system in place at most (especially as implemented at large) graduate schools discourages entry into "fringe" areas such as geometry. Thus, in a certain very real sense, the study of geometry has not been in the mainstream of the training of professional mathematicians: those majoring in mathematics in college and going on to pursue doctoral studies in graduate school.

Since there are not enough individuals who call themselves geometers to go around, most survey courses in geometry are taught by individuals with a narrow base of geometrical knowledge. Such individuals rely heavily on the geometry texts in print in teaching the Survey Course since teaching a course based on readings and on their own knowledge base imposes a heavy preparation burden. (Geometry Courses are taught by the one member of the department who got the course listed in the catalogue in the first place, are taught by a "draftee," or fall into disuse.

As noted before, many parts of mathematics have been developed in geometric form. Furthermore, a true renaissance of geometry has occurred in recent years. Examples of this ferment in geometric ideas include: the development of a new branch of mathematics, computational geometry; exciting breakthroughs in understanding the geometric structure of space (with resulting heavy cross fertilization with workers and ideas in mathematical physics); breakthroughs in the study of the mathematics involved in tiling problems for both the plane and higher dimensional spaces; an explosion of geometric ideas related to the theory of graphs with application to many areas of mathematics and operations research; dramatic new developments in the theory and application of the theory of knots; exciting connections between developments in the theory of dynamical systems and the geometry of sets (fractals); dramatic uses of geometrical methods in image recognition and processing; and use of geometric methods in the control and motion planning for robots and robot arms, to mention but a few of the most visible examples. This listing could easily be extended. Hence, it is increasingly unfortunate that both teachers (already teaching and new ones being trained) and future researchers have not had available to them a vehicle for being exposed to the exciting new developments in geometry. Though geometric thinking itself may not be taught as part of the mathematical mainstream, geometry and geometric thinking is "infiltrating" mainstream mathematics more than ever before.

Geometry and Teacher Training

If American citizens are not to be raised as geometric illiterates, teachers in our grade schools and high schools will have to be broadly trained geometrically themselves. We have already examined the trend that new high school mathematics teachers entering our schools are few in number and have had less opportunity to be exposed to geometry than high school teachers of earlier generations.

Many experiments are now being conducted to try to develop specialists to teach mathematics K–6. The need for "mathematics specialists" has been raised by the resistance of traditionally trained K–6 teachers to new developments and teaching methods in grade school. (Traditionally trained teachers in elementary school usually take a single course in mathematics as part of their teacher preparation. This course concentrates almost exclusively on the development of thinking about the base 10 number system, associated problems in addition, subtraction, multiplication, and division, and on measurement. This course rarely mentions any ideas in the area of geometry beyond simple taxonomy of simple shapes.) Emerging programs that urge specialists for elementary school to major in subject areas in college, as more reasonable preparation for teaching in grade school, will wind up subjecting such students to the very narrow type of geometry course now taught as a Survey Course in our colleges. One of the few positive trends to note is that many teachers, both those planning to teach in high school or pre-high school environments, are being forced or encouraged to study the computer language called *Logo.* Creative use of the *Logo* language can permit students to be exposed to a wide range of open-ended, exploratory experiences with geometry.

Clearly, the Survey Course in geometry will play a large role in the exposure of future teachers to geometry. This is likely to become more so if future grade school mathematics specialists take this type of course. Thus it seems both wise and necessary for the mathematics community to significantly revamp the Survey Course. Such a change will be a service not only for future teachers and their students but for future researchers as well.

Goals in Changing the Survey Course as it Currently Exists

In attempting to change the content of the Survey Course, there are a variety of reasonable goals. Among these is the possibility of significantly changing the content of what is taught in high school by giving future high school teachers preparation in the geometry that might be part of a future high school geometry curriculum. Another goal is to encourage larger groups of students with interests in areas related to mathematics (e.g., computer science and engineering) to explore the many advantages that would accrue to them in being more broadly versed in geometric ideas. (The self-contained and quick starting nature of geometry makes this feasible.) A final goal might be to provide a rich variety of geometric concepts and tools for future research mathematicians both in traditional as well as emerging areas of mathematics, and to encourage more future research mathematicians to work in the area of geometry by exposing students to easily accessible unsolved problems.

Benefits of a Newly Constituted Survey Course

Although clearly geometry deserves to be studied for its own sake, many important objectives of mathematics study in general can show from studying geometry. Below is a partial list of some of the benefits of a revised geometry Survey Course (listed in random order):

- To show how geometric mathematics is affecting modern life (i.e., compact disk recorders, CAT scans, HDTV [high definition TV], image processing, richer understanding of the geometry of space, robots, new types of maps, etc.).
- To encourage visual thinking and reasoning (use of diagrams and models as modes of thought and problem solving).
- To learn the interplay of pure and applicable ideas (e.g., error-correcting codes and sharp pictures of Uranus and Jupiter, know theory to study DNA, etc.).
- To learn the distinction between the mathematics of geometry and the geometry of physical space.
- To show the rich history of geometry as a subject and the connection between geometry and other disciplines outside of mathematics such as philosophy and physics.
- To show how computers and specific software environments can be an aid to geometric thinking.
- To foster better writing, verbal, and communication skills when dealing with technical ideas.
- To illustrate how ideas in mathematical modelling are of value in a geometrical setting, and how geometric thinking is a tool for the mathematical model builder (i.e., use of graph theory to study problems in making deliveries to discrete locations, say oil to homeowners).
- To learn how ideas developed for one application of mathematics are often transportable to other situations (e.g., getting a fire truck to a fire quickly and designing efficient paths for robots in a workspace).
- To obtain experiences in problem posing and problem solving.
- To illustrate domains in which experiments can be done in mathematics and have students carry out such experiments (e.g., soap bubbles, tilings, mirrors to study symmetry).
- To expose students to a variety of unsolved problems in geometry.
- To learn how one part of mathematics makes contributions to other parts (e.g., the interplay between algebra and geometry, and combinatorics and geometry).
- To illustrate how basic concepts such as distance, function, volume, etc., are of use in a geometric setting.
- To illustrate the power of abstraction, special cases, and the use of symbolism.
- To learn what a mathematical proof means and to give examples of such proofs. (Note: there is no reason, however, to restrict the domain of such proofs to theorems that appear in Euclid or similar results.)

Content for a New Survey Course

In attempting to design a new Survey Course in Geometry a variety of principles could be applied. Among these are that basic geometrical concepts and methodologies should be represented, that modern applications should be shown, that breadth as well as depth be respected, that a variety of geometric proof techniques be shown, and that a variety of different types of geometrical objects be examined. In addition to teaching a course based on significantly new content, I believe that the mathematics community should take advantage of new computer technologies (computer environments such as *Logo* or *Mathematica*) and the use of videotape. For example, many applications of geometry are best introduced to a student in visual form using videotape rather than in written form. Appendix II shows various ideas for development of a video applications library to support existing and future text materials used in the teaching of geometry.

As a brief perusal of Malkevitch [8] quickly reveals, an exhaustive look at geometry in a semester sequence is not realistic. There is just too much attractive and important material. Any specific geometer is likely to have a somewhat different collection of topics and ordering for teaching these topics for a survey course from another geometer. However, I believe there is widespread agreement that the current course must be changed, moved in a direction away from axiomatics, and that any new course have a "core" of principles and content. In Appendix I, I have listed one of many possible approaches to both the content and organization of a new survey course that I have considered. Implementation of such a course will, I believe, be a major step toward attaining greater geometric literacy for teachers, the lay public, and mathematicians as well.

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Appendix I: Geometry Tomorrow

Outline of some of the major topics to be covered in a geometry course of the future (listed in random order):

- Combinatorial ideas vs. metric ideas
- Convexity
- Geometry of physical space (relation to axiomatics)
- Graph theory ideas
- Computational geometry ideas
- Symmetry, polyhedra, and tilings
- Visual thinking
- Area and volume (Bolyai-Gerwin, Hadwiger, Banach-Tarski)
- Dynamical systems and fractals
- Differential ideas
- Applications

- Role of dimension
- Proof tools: induction, infinite descent, examples, constructions, divide and conquer, etc.
- Isomorphism concepts
- Geometric transformations
- Digital geometry
- Packing and covering problems
- Lattice point problems

Assumed prerequisites: year of calculus or principles of mathematics course, and knowledge of matrix notation and multiplication (but not necessarily of linear algebra).

Unit I

A. What is Geometry?
   i. Geometric pearls
      a. Bolyai-Gerwin theorem
      b. Euler's traversability theorem
      c. Art gallery theorem (Fisk's proof)
      d. Helly type theorems
      e. Curves of constant breadth
      f. Distance realization problems
      g. Euler's polyhedral formula
      h. Penrose tiles
      i. Pick's theorem (lattice points)
      j. Desargues' theorem
   ii. Visual thinking
      a. Value of drawing diagrams
      b. Value of constructing models
      c. Geometry experiments (soap bubbles, etc.)
      d. Computer environments

B. Different Approaches to Geometry
   i. Difference between metric geometry and combinatorial geometry
   ii. Axiomatic geometry and the geometry of physical space
   iii. Isomorphism concepts
   iv. Historical role of parallelism
      a. Two space
      b. Three space
      c. Four space
      d. Space-time
      e. Surfaces embedded in three space
      f. Dimension
   v. Deductive vs. inductive approaches to the study of geometry
   vi. Geometry and the computer
   vii. The relation of geometry to algebra and other parts of mathematics

C. Proof Tools of the Geometer
   i. Induction
      a. On number of objects
      b. On dimension
   ii. Infinite descent
   iii. Constructions
   iv. Algebra
   v. Arguments based on symmetry

Unit II

A. Types of Geometric Structures (Graphs, Planes, Spaces, Block Designs, Convex Sets)

B. Graph Theory
   a. Traversability
   b. Trees
   c. Coloring problems
   d. Planarity
   e. Matchings
   f. Network algorithms (shortest paths, flows, minimum-cost spanning trees, etc.)

C. Planes
   a. Affine planes
   b. Projective planes
   c. Hyperbolic planes (infinite and finite examples)
   d. Role of Desargues' "statement"

D. Space
   a. Euclidean, projective, hyperbolic space
   b. Axiomatics and geometry of space

E. Block Designs

F. Convex Sets
   a. Helly, Radon, and Cartheodory's theorems
   b. Minkowski addition
   c. Curves of constant breadth
   d. Geometric inequalities (isoperimetry)
   e. Lattice point problems
   f. Packing and covering problems

Unit III

A. Geometrical Transformations (viewed not as an approach to the theorems of Euclidean geometry, but for their own sake)

B. Transformations and Their Relationship to Space

C. Transformations and Their Relationship to Metric Properties (i.e., congruence)

D. Geometric Transformations Viewed Geometrically

E. Geometric Transformations Viewed Algebraically

Unit IV

A. Symmetry and Regularity Polygons
   a. Plane polygons
   b. Convex polygons
   c. Self-intersecting polygons
   d. Packing and covering problems

B. Tilings
   a. Tilings with regular polygons
   b. Tilings with convex polygons
   c. Symmetry properties of tilings
   d. Aperiodic tilings
   e. Penrose tilings

C. Polyhedra
   a. Regular polyhedra
   b. Archimedean polyhedra
   c. Combinatorial properties of polyhedra
      i. Euler's formula
      ii. Steinitz's theorem
   d. Minkowski addition aspects of polyhedra
   e. Graphs of polyhedra
   f. Tilings in space

D. Symmetry Groups
   a. Symmetry groups of tilings, patterns, fabrics, etc.

Unit V

A. Area and Volume
B. Equidecomposability
C. Role of Archimedes' Axiom
D. Squaring the Circle
E. Banach-Tarski Paradox
F. Dynamical Systems and Fractals

Unit VI

A. Computational Geometry
B. Triangulations
C. Voronoi Diagram
D. Sweep Line Methods
E. Convex Hull
F. Principles of Design for Geometric Algorithms

Unit VII: Topological Ideas

A. Geometry of Surfaces
   a. Orientability (Moebius Band)
   b. Torus
   c. Klein Bottle

B. Knots
   a. Geometric transformations of knots
   b. Classification of knots

Unit VIII: Geometric Optimization Problems

A. Linear Programming
B. Isoperimetry
C. Packing and Coverings
D. Network Optimization

Unit IX: History of Geometry

A. Geometry in the Ancient World
B. Geometry During the Renaissance
C. Geometry Up to the 20th Century
D. Geometry in the 20th Century

Note: There should be biographical material about the great contributors to geometry, including, where possible, portraits or photographs.

Unit X: Applications of Geometry

Applications should probably be sprinkled in and included in an integral manner with the other parts of the materials being developed. However, here are some particularly topical areas that might be mentioned (see Appendix II for additional examples):

A. Robotics
B. Computer Vision
C. Computer Graphics
D. Solid Modelling
E. Operations Research

Note 1: Unsolved problems in geometry would be mentioned throughout the course.

Note 2: For bibliographic references in support of a wide variety of classical and recent topics, see Malkevitch [8].

Appendix II: Ideas for a Videotape

Edge Traversal

Situations:
   Curb inspecting
   Street sweeping
   Garbage collection
   Mail delivery
   Advertising circular delivery
   Painting line down center of roads
   Snow removal
   Parking meter collection and enforcement
   Police or museum guard patrol routes
   Pipe, wiring, or duct inspection

Mathematics:
   Graphs as models
   Euler's traversability theorem
   Chinese Postman Problem
   Johnson and Edmond's algorithm
   Deadheading and repeated edges

Practitioners:
   U.S. Postal Service
   Sanitation Department
   Department of Parking Enforcement
   University Operations Research Departments (MIT, Maryland, Stony Brook)
   AT&T Bell Laboratories; Bell Communication Research

Vertex Traversal

Situations:
   Meals on wheels
   Deliveries to supermarkets, restaurants, etc.

Garbage pickup from industrial sites
Machine inserter schedules
Computer solution of jigsaw puzzles
School bus routes
Camp pickup routes
Parcel post delivery and pickup
Pizza delivery
Special delivery of mail
Pickup of coins from pay telephone booths

Mathematics:
Graphs as models
Hamiltonian circuits in graphs
Traveling salesman problem
Asymmetry of costs
Complexity
K-opt methods
Greedy algorithms
Vehicle routing problems
Clarke-Wright algorithm

Practitioners:
Sanitation Department
U.S. Postal Service
Federal Express
Parcel Post
School Boards
Camps
University Operations Research Departments (MIT, Stony Brook, Maryland)
AT&T Bell Laboratories; Bell Communications Research

Voronoi Diagrams

Situations:
District planning
Drainage regions
Market structure (anthropology)
Robot motion planning

Mathematics:
Computational geometry
Perpendicular bisector
Convex set
Convex hull
Concurrence, concyclic points
Line sweep algorithms

Practitioners:
University Mathematics and Computer Science Departments (Smith College, Courant Institute, University of Illinois, Princeton, Rutgers)

Robots (Motion Planning)

Situations:
Industries which employ mobile robots
Planetary surface exploration

Mathematics:
Graphs as models
Visibility graphs
Shortest path algorithms
Minkowski addition
Parallel domains
Vision

Practitioners:
General Motors, Ford, Chrysler, etc.
Universities (MIT, Yale, Courant Institute (NYU), Stanford)
AT&T Bell Laboratories

Note: Other aspects of robotics also involve geometrical ideas. These include the local motion planning of the gripper of a stationary robot.

Bin Packing

Situations:
Machine scheduling (independent tasks)
Organizing computer files on disks
Advertising breaks
Want advertisements in newspapers

Mathematics:
Packing problems
Heuristic algorithms
Measures of efficiency
Time space tradeoffs
Complexity
Simulation

Practitioners:
Operations Research Departments (Berkeley)
AT&T Bell Laboratories

Distances

Situations:
Car travel
Urban distance
Biology (evolutionary trees)

Mathematics:
Taxicab metric
Abstract properties of distance
Sequence comparison
Levenshtein distance

Practitioners:
AT&T Bell Laboratories

Shortest and Longest Paths

Situations:
Fire truck and ambulance routing
Building construction
Space program (flight planning)
Robot motion planning

Mathematics:

Graphs, digraphs, and weighted graphs and digraphs
Dijkstra's algorithm
Critical path method

Practitioners:
Operations researchers

Minimum-Cost Spanning Trees

Situations:
Synthesis of communication networks
Road planning

Mathematics:
Graphs as models
Trees
Spanning trees
Kruskal's algorithm
Prim's algorithm
Greedy algorithm

Error Correcting Codes

Situations:
Compact disk players
Computer codes
Space programs
HDTV

Mathematics:
Binary sequences
Distance (Hamming distance)
Matrices
Information content

Practitioners:
Compact disk manufacturers (Philips)
Universities (California Institute of Technology, MIT)
AT&T Bell Laboratories

Coloring Problems

Situations:
Scheduling committees, final examinations, railroads
Fish tanks and animal confinement patterns
Maps
Placement of guard in art galleries

Mathematics:
Graphs as models
Vertex colorings
Face colorings
Edge coloring
Complexity

Practitioners:
Universities with graph theory specialists
AT&T Bell Laboratories; Bell Communications Research

Data Compression

Situations:
Image transmission and storage
Text transmission and storage

Mathematics:
Binary numbers
Digitalization of text and images
Huffman codes
Fractal methods

Practitioners:
Universities (MIT, Georgia Institute of Technology)
AT&T Bell Laboratories; Bell Communications Research
NASA

Geometric Transformations and Symmetry

Situations:
Computer graphics
Analysis of fabrics
Analysis of archeological facts
Cartography
Analysis of art (Escher paintings)

Mathematics:
Group theory
Functions and transformations
Strip groups
Wallpaper groups
Color symmetry

Practitioners:
University mathematicians

Unfolding Polyhedral Surfaces

Situations:
Catching a spider on the wall of a cube
Drawing a map of a spherical surface
Unfolding the surface of the brain
Layouts for packages

Mathematics:
Development of polytopes
Projection mappings
Distance on polyhedral surfaces

Block Designs

Situations:
Drug testing
Agricultural productivity
Scheduling workers
Scheduling tournaments

Mathematics:
Finite geometries
BIBD's
Orthogonal Latin Squares

Practitioners:
- Universities (Ohio State University, CAL Tech)
- AT&T Bell Laboratories

Mathematical Programming

Situations:
- Blending gasolines
- Blending juices
- Manufacture of processed foods
- Scheduling
- Shipment of goods
- Vehicle routing
- Hospital management
- Portfolio management

Mathematics:
- Linear programming
- Integer programming
- Linear inequalities
- Solution of liner equations
- Network flows
- Transportation problem

Practitioners:
- Universities (Rutgers, Princeton, Stony Brook)
- AT&T Bell Laboratories
- Oil companies, airlines, car companies, defense industries

Art Gallery Theorems

Situations:
- Surveillance in museums, banks, and military installations

Mathematics:
- Convex sets
- Types of polygons
- Triangulations
- Colorings

Practitioners:
- Computational geometers

Euclidean Geometry

Situations:
- Length of carpet remnants
- Time remaining on a partially used tape

Mathematics:
- Geometry of the circle (circumference)
- Areas and perimeter concepts
- Isoperimetry

Alternate Organization

The situations above are organized by mathematical theme. Other approaches also exist, in particular, showing applications of geometry to a particular subject area. Several examples of this are given below:

Applications of Geometry to Business:
1. Traversability problems
2. Minimum-cost spanning trees
3. Facility location problems
4. Coloring problems (scheduling problems)

Applications of Geometry to Medicine:
1. CAT scanners and other medical imaging systems
2. Kidney stone machines
3. Brain mapping studies

Applications of Geometry to Biology:
1. Structure of the gene (intersection graphs, interval graphs)
2. Food chains, niche spaces, competition (intersection graphs)
3. Ecology (fractals)
4. Shape of biological forms (isoperimetry)

Applications of Geometry to Chemistry:
1. Quasicrystals (Penrose tiles, crystallography)
2. Dynamics of chemical reactions (dynamical systems)

Applications of Geometry in Communications:
1. Synthesis of communication networks
2. Vulnerability of communication networks
3. Phone exchange systems
4. Error-correction methods and codes
5. Data compression (compression of text and images)
6. Digitalization of images
7. Image processing (filtering, etc.)

Applications of Geometry in Social Science:
1. Analysis of fabrics, designs, and pottery (anthropology), groups, symmetry patterns
2. Kinship systems (anthropology), graph theory
3. Mobility (sociology), Markov chain digraphs
4. Equilibrium analysis (economics), dynamical systems