RECOMMENDATIONS ON UNDERGRADUATE MATHEMATICS COURSES INVOLVING COMPUTING

A Report of

The Panel on the Impact of

Computing on Mathematics Courses

October 1972

TABLE OF CONTENTS

1. Preface

The growing influence of modern electronic computing in many fields of knowledge has contributed to a dramatic increase and diversification in the application of mathematics to other disciplines. No longer are the uses of mathematics confined exclusively to the physical sciences and engineering; they are found with increasing frequency in the social, behavioral, and life sciences as well. Correspondingly, the use of the computer has led to different requirements for the solution process in mathematics itself. Theory construction and model building have assumed a different dimension; in addition to knowing existence theorems, the user of mathematics must know constructive methods for solving problems, and he must have the means to ascertain the efficiency as well as the correctness of these methods.

These developments have created new challenges with regard to revision of the undergraduate mathematics curriculum. The basic curriculum should reflect the contemporary points of view associated with computer application in mathematics; it should acquaint the students with the newly developed methods of solving standard problems and also introduce them to the host of problems which have arisen in the past few years.

There now appears to be growing recognition that more consideration should be given to the potential impact of the computer on the basic undergraduate courses which serve not only potential mathematicians and computer scientists but many other students as well. The present report is the result of the first study of this problem by CUPM.

Although a consensus about the role of computers in the basic mathematics curriculum has not yet evolved, we believe there is an urgent need for experimentation in this area. This report presents ideas for such experimentation by proposing changes in various basic mathematics courses and by suggesting some new courses which are designed to take advantage of the presence of computers.

As is the case with all CUPM reports, these recommendations must be regarded as general suggestions which will need to be adapted to local circumstances and revised in the light of subsequent experience. Nevertheless, mathematics departments should immediately concern themselves with the ideas outlined in this report so that they can prepare their students for the uses of mathematics in the context of the availability of computers.

2. Premises of this Study

We suggest four ways in which the computer can influence undergraduate mathematics education:

(i) Computing can be introduced into traditional mathematics courses;

(ii) New courses in computationally-oriented mathematical topics can be designed;

(iii) The entire curriculum can be modifed to integrate computing more fully into the student's program;

(iv) Computers and computer-related devices can be used as direct aids to mathematical instruction.

This report addresses itself to (i) and certain aspects of (ii). As for (iii), some possible curriculum restructuring relating to computing has been discussed in the CUPM report Recommendations for an Undergraduate Program in Computational Mathematics. Finally, (iv) is a very broad area which would require a separate study of its own; we include only a brief discussion of some related topics in the final section of the report.

We do not suggest that all mathematics instruction be modified along any of these lines. In recent years it has become appropriate to speak of the mathematical sciences in a broad sense rather than of mathematics in the more familiar, narrower sense. This situation indicates a need for different avenues within mathematics education; the introduction of computer-oriented material should therefore be regarded as a development parallel to the standard curriculum which interacts with the standard curriculum at a number of places.

Nearly every student taking mathematics courses can benefit from some computer-oriented mathematics instruction. The use of computers is beginning to pervade all phases of life in our society, and in most disciplines, including mathematics, there is a need for students to become familiar with some aspects of computing. Many mathematics departments have observed that well over half of their undergraduate majors enter computer-related careers or graduate programs after graduation. Clearly, these students would benefit considerably from computer-oriented courses and curricula. Computer-oriented courses also serve all those students from other disciplines who are interested in learning more mathematics in order to solve problems from their own fields. Modern applied mathematics has a strong computer orientation; when students enter this field, those whose education stresses concern for computational problems have a decided advantage over those who are familiar only with theoretical results. Finally, the growing trend toward introducing computers in high schools will require that prospective teachers learn about the interaction between the computer and mathematics.

Although the content and objectives of computational material differ considerably for various groups of students, computing provides all of them an unusual opportunity for active participation. For this reason the motivational aspects of computing are significant for most students, and the value of such motivation should not be underestimated.

A more substantive objective must be to select course material and approaches so as to reflect the actual influence of computing on mathematics. The recommendations which follow are based on the premise that any program which seeks to reflect this influence should stress four points--namely, algorithms, approximations, model building, and the nature of the entire problem-solving process.

Algorithms. The modern computer's development as a general-purpose problem-solving system derives not so much from its arithmetic capabilities but from its ability to handle logical and nonnumerical problems. From a mathematical viewpoint this has led to a greater emphasis on the construction and analysis of algorithms for the actual solution of mathematical problems rather than only on the proof of the existence of solutions. Stressing the algorithmic aspects forces the student to state both the problem and the method of solution in precise and unambiguous terms. It fosters his ability to organize and formulate logically an attack on a problem as well as to recognize and clarify the assumptions he is making in order to solve the problem.

Approximations. In most analysis courses numerical algorithms are more prevalent than nonnumerical ones. This leads to questions of error or, more generally, to questions about the quality of the approximations produced by the algorithm. If an algorithm produces an answer, some statement is needed to relate this answer to a solution of the original mathematical problem. If a process appears to converge, there is a need to prove that the process converges as well as to determine how rapidly it converges. If a method is bound to be applicable to certain input data, it is necessary to establish what happens when changes are introduced in these data. Clearly, in undergraduate courses these questions can rarely be answered satisfactorily but the student should acquire a concern for them and an appreciation of their importance.

Model Building. An important part of every real application of mathematics is the recognition and formulation of a satisfactory mathematical model of the given nonmathematical problem. Developing the student's skill in this process should be an objective of every course involving computer applications.

The Problem-Solving Process. Modeling, the development of algorithms, the study of the approximations used, and the computation and interpretation of results are all principal steps in the process of solving a problem on a computer. It is important to stimulate in the student an understanding of this process viewed as a whole by discussing and assigning the complete solution of appropriate simple problems.

The four points raised here--namely, the stress on algorithms, the development of a concern for the quality of approximations, the emphasis on model building, and a general emphasis on the entire problem-solving process--should be considered as general objectives in the student's program. In those undergraduate courses which involve a limited amount of computing, little more can be done than to illustrate the importance of these points and to instill in the student an intuitive understanding and concern for them. This requires a careful selection of the computational topics to be discussed and of the problems to be assigned. In other words, without underestimating the motivational value of computing, we believe that the introduction of computational material into a mathematics course should go beyond merely illustrating a mathematical concept. It should at least provide a definite answer to a specific question or problem in order to give the student deeper insight into the theory, the model, and the algorithms used.

Implementation and Precautions. There are wide variations in the extent and type of computer use which can be introduced into a traditional mathematics course. Such variations arise both from differences in computing facilities and from differences in the instructors' opinions of how essential these uses can and should be for the course. Where computing facilities are readily available or where courses are modified extensively to emphasize the four objectives discussed above, the trend is often to use the computer frequently and in a matter-of-fact manner. This means that computer-related material is presented throughout the course and that computer problems are assigned as a regular part of the homework; the student is expected to master these problems in order to have a coherent understanding of the subject. Where computing facilities are not so readily available or where computer-related material enters the course only in a secondary, supportive role, the trend is to consolidate computer use into the solution of a number of relatively substantial problems and to expect the student to apply his mathematical knowledge to these problems, but not to demand mastery of these problems for a coherent picture of the course. In this case, extra credit is sometimes given for the computer component of a course.

In whatever way the computer is used, there are a number of precautions which ought to be observed. Primarily, one should neither misuse nor overuse the computer. The computer is certainly misused when one is not mathematically honest about what it can or cannot do. For example, a computer can approximate a limit, but it cannot "compute" one or verify its existence, nor can it "test" a function for continuity. Specific examples of overuse of the computer are harder to provide, but it can be recognized when computing begins to crowd mathematical material out of the course or when students become bored by it. Overuse of the computer can result when the excitement of the new approach obscures the principal purpose: to teach mathematics. It is primarily the algorithmic approach together with the other three objectives, rather than the actual use of a computer, which will help to advance this purpose. Many points about algorithms can be made without using a computer at all; three-digit arithmetic can be used to discuss approximations and roundoff error, and model-building and problem-solving expertise can also be gained from judiciously chosen paper-and-pencil problems. Also, students need not program every algorithm they encounter, and experiments with preprogrammed algorithms can often provide more insight than the lengthy drudgery of debugging a complicated program. It is up to the individual instructor to maintain a proper balance between the use of the computer and the other components of the course.

3. Basic Courses

3.1 Introduction

In this section we discuss five one-semester beginning undergraduate courses which include an emphasis on computing. As in all CUPM reports, the outlines given here are not meant to be prescriptive but are intended to extend the exposition of our ideas in the previous section by giving suggestions and possible approaches for implementing them.

For reference purposes we begin with a list of brief catalog descriptions for these courses. The descriptions do not include any programming requirements; these are discussed in Section 5.2.

MC-0. Elementary Functions and Problem Solving. [Prerequisite: College admission] Basic computer programming, elementary functions, matrix operations. These topics are to be motivated by, and applied to, practical problems.

MC-1. Calculus I with Computer Support. [Prerequisite: MC-0 plus trigonometry, or equivalent mathematical background] Differential and integral calculus of the elementary functions with associated analytic geometry, supported by computer applications.

MC-2. Calculus II with Computer Support. [Prerequisite: MC-1] Techniques of integration, introduction to multivariate calculus, and elements of differential equations, supported by computer applications.

MC-DM. Discrete Mathematics. [Prerequisite: No specific course prerequisite, but see page 590] Concepts and techniques in discrete mathematics that find frequent applications in computing problems.

MC-3. Algorithmic Elementary Linear Algebra. [Prerequisite: MC-0 or equivalent background] An introduction to matrix and vector algebra in n dimensions with an emphasis on algorithmic aspects.

The four courses MC-0 through MC-3 represent computer-oriented versions of the courses Mathematics 0 through 3 in the 1972 CUPM report Commentary on a General Curriculum in Mathematics for Colleges (GCMC Commentary). In the case of MC-0 and MC-3, the material in the GCMC Commentary courses was considerably modified and rearranged in order to introduce a fairly strong emphasis on computation. In MC-1 and MC-2, on the other hand, the purpose and the outline have remained essentially the same as for traditional courses; the emphasis on what is taught, however, has shifted along with the shift in the kinds of applications that are possible with the computer. The remaining course MC-DM represents a new development. It has a strong algorithmic flavor and introduces material of considerable importance in many computer applications. The course may not only supplement the standard curriculum but could also serve well as a first mathematics course for students from many disciplines.

Ideally, a student entering any of these computer-related courses other than MC-0 should have at least a rudimentary knowledge of programming. Since this is, at present, an unrealistic requirement, several possible alternatives are suggested in Section 5.2. Since these alternatives depend strongly on local circumstances, no further mention of them is made in the outlines. Only in the case of MC-0 is some time allotted to introduce certain elementary computer concepts.

In each of the following outlines, the suggested pace is indicated by assigning a number of hours to each group of topics. A standard semester contains 42 to 48 class meetings, and we follow the GCMC Commentary in allowing approximately 36 hours for discussion of new material; the remaining time can be devoted to tests, reviews, etc.

3.2 Course Outlines

MC-0. Elementary Functions and Problem Solving

[Prerequisite: College admission] The aim of this freshman-level course is to teach students ways to approach problems in the physical, natural, and social sciences and to equip them with some fundamental mathematical and computational tools for the solution of these problems. Typical problems are concerned with measurement and prediction: given a process such as a factory producing steel, traffic moving on a city street, or a shifting population, it is desired to predict future properties of the process on the basis of past measurements. The approach taken in the course is first to have students model and simulate specific processes using a computer and then look for functional relationships between various aspects of these processes, e.g., between time and the total output of steel, between traffic light timing and traffic density, or between past and future population distributions. The objectives of this approach are to create an understanding of modeling through approximation and simulation and a feeling for the types of questions asked about models and functions.

Studies of elementary functions, computational techniques, and matrix operations are interwoven in the course and are used to illustrate and motivate one another. Depending upon the selection and treatment of particular topics, this course may serve either as a refresher course for students going on to calculus or as a terminal course for students who intend to take only one course in mathematics. However, this course alone probably will not prepare students adequately for a one-year sequence in calculus, since it does not contain the topics from trigonometry which they will need. The orientation towards applications and the emphasis on computing should make the course attractive to many students who might otherwise avoid more traditional mathematics courses. There are no prerequisites, as instruction in the use of a computer is integrated with the rest of the course.

COURSE OUTLINE

1. Introduction. (6 hours) Number representation, algorithms, elements of programming, functions, relations.

2. Linear and quadratic functions. (8 hours) Simulations involving constant and accelerated rates of change, graphs of linear and quadratic functions, zeros, maxima, minima, applications.

3. Linear programming. (4 hours) Linear functions of two variables, linear inequalities, maxima, minima, applications.

4. Matrix operations. (6 hours) Representations of tabular data, subscripts, matrix and vector operations, simultaneous equations, applications.

5. Algebra of functions. (6 hours) Algebraic operations on functions, polynomial and rational functions, maxima, minima, zeros, inverses, composition.

6. Exponential and logarithmic functions. (6 hours) Simulations of exponential growth, properties of exponents, logarithms as inverses of exponentials.

COMMENTARY

1. Introduction. Simple mathematical concepts can be introduced or reviewed in the context of teaching the rudiments of programming in a computer language such as APL, BASIC, or FORTRAN. For a start, students should learn to use arithmetic, branching, and simple looping statements; other techniques, such as subroutines, can be considered later as the need for them arises. Machine arithmetic can be contrasted with ordinary arithmetic, with examples of roundoff error being given.

Functions should be introduced as single-valued rules of association, with the relationships between the inputs and outputs of computer programs providing many examples of both numeric and nonnumeric functions. Questions of scaling which arise in the development of a simple program for graphing functions can be used as a bridge to the next section on linear functions.

2. Linear and quadratic functions. In studying rates of change, the student can first write a computer program to model a situation involving constant change. After this model has been used to motivate a study of linear functions, the computer program can be modified by the addition of a single statement to model constant acceleration, thereby motivating a study of quadratic functions; later, the added statement can be changed to have the program model more complicated rates of acceleration (e.g., a bouncing ball or exponential growth). Questions about zeros, maxima, and minima should be raised and answered to ascertain properties of models, functions, and graphs. In this way the study of linear and quadratic functions provides a framework for later material in the course.

In addition to using the computer for simulation, one can stress the algorithmic aspects of graphing by using programs to compute the slopes of lines or the zeros of a quadratic function by the quadratic formula. Zeros, and in particular square roots, can also be approximated by the bisection method.

3. Linear programming. The study of linear functions leads naturally to a study of linear programming in two dimensions. Boundary conditions lead to a consideration of linear inequalities and to

the solution of simultaneous equations in two unknowns in order to determine constraint regions. Cost functions can be introduced as functions from vectors in those regions to numbers, and the location of the maxima and minima of these functions at the vertices of regions can be demonstrated by drawing level curves.

4. Matrix operations. As a further example of the applicability of linear methods, models of population movement can be studied. One can introduce a vector V to represent the population distribution at a given time and a matrix M to represent the percentage redistribution of population over a year's time. Matrix and vector multiplication can be motivated by writing a computer program to model population movement over a number of years and observing that M^nV is the population distribution after n years. Finally, this model can be used to motivate the solution of simultaneous linear equations or the inversion of a matrix to find the equilibrium distribution.

5. Algebra of functions. By associating functions with subroutines which compute them, one can motivate a general discussion of the domains and ranges of functions, as well as of algebraic operations on functions. Applied to polynomials, this leads naturally to the rational functions. In order to answer standard questions about these functions, one can discuss numerical techniques such as bisection and hill-climbing for locating zeros, maxima, and minima. The inverse of a function can be found by computing the zeros of translated functions.

6. Exponential and logarithmic functions. Computations involving population growth, interest rates, or radioactive decay lead to a study of exponential functions. The logarithm can be computed by the method developed in Section 5, and its properties can be established from the properties of exponentials.

REFERENCES

No presently available text is suitable for this course. While some of the references listed below contain material that can be used in the course, no text develops computing and mathematics together along the lines suggested by units 1 and 2. The approaches to computing in two of the references bracket the suggested approach:

Vogeli, et al., is generally too elementary and does not use computing in a substantial way, while Higgins presumes too much prior experience both in computing and in mathematics. The remaining references are programming texts which contain some examples appropriate for the course.

Barrodale, Ian; Ehle, Byron; Roberts, F. D. K. Elementary Computer Applications in Science, Engineering, and Business. New York, John Wiley and Sons, Inc., 1971.

Gruenberger, Fred and Jaffray, George. Problems for Computer Solution. New York, John Wiley and Sons, Inc., 1965.

Higgins, G. Albert. The Elementary Functions: An Algorithmic Approach. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1974.

Kemeny, John G. and Kurtz, Thomas E. Basic Programming, 2nd ed. New York, John Wiley and Sons, Inc., 1971.

Maurer, H. A. and Williams, M. R. A Collection of Programming Problems and Techniques. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1972.

Vogeli, Bruce R.; Prevost, Fernand; Gilbert, Glenn; Carroll, Edward. Algebra One. Morristown, New Jersey, Silver Burdett Company, 1971.

MC-1 and MC-2. Calculus I and II with Computer Support

[Prerequisite for MC-1: MC-0 and trigonometry, or equivalent background; prerequisite for MC-2: MC-1] The introductory courses on calculus appear to be those mathematics courses in which the use of the computer has been most popular. One reason for this is the fact that many concepts and methods in the calculus have a practical flavor which can be enhanced by introducing computing. While the motivational value of computational work plays a considerable role, the student can also handle more realistic problems using the computer as a tool and with it learn to appreciate more fully the power and usefulness of calculus.

There are at this time no firm guidelines as to how the computer should be introduced into calculus courses; many radically different experiments have been conducted and are still being carried on. We believe that at present a practical and rather attractive approach is to use the computer to support courses which are more or less traditional in the selection and sequencing of the material. Hence the courses described below are, at least in outline, identical with the courses Mathematics 1 and 2 in the GCMC Commentary, and their basic purpose remains essentially the same--namely, that of being an intuitive, yet sound, introduction to limits in various forms, such as derivatives, integrals, or sums of series, along with applications of several types, such as maximum-minimum problems or questions leading to integrals.

We do not believe that it is enough to teach the calculus courses more or less as usual and to assign computer projects as supplements to the course. Such an approach does not take full advantage of the interplay between theoretical and algorithmic ideas. The new courses must be taught in a different manner, if only because computing can provide much useful motivation for the calculus. Moreover, the emphasis on what is taught should shift towards the kind of realistic applications that are possible with the computer. In these ways the computer can be used to support the presentation of material rather than merely to supplement it.

The commentaries below indicate ways in which this supportive role of the computer can be accomplished. For the most part these commentaries are meant to supplement rather than replace those in the GCMC Commentary.

COURSE OUTLINE FOR MC-1

1. Introduction. (4 hours) Review of the function concept. Function evaluation and graphing on a computer.

2. Limits, continuity. (3 hours) Limit and approximation defined intuitively. Derivatives as examples. Definition of continuity, types of discontinuity, Intermediate Value Theorem. Computational applications involving the bisection method and showing effects of truncation and roundoff error.

3. Differentiation of rational functions; maxima and minima. (5 hours) Computational projects involving search algorithms for finding extrema. Newton's method.

4. Chain rule. (3 hours) Include derivatives of functions defined implicitly, inverse function and its derivative. The algorithmic aspect of functional composition.

5. Differentiation of trigonometric functions. Higher derivatives. (3 hours)

6. Applications of differentiation. (3 hours) Tangent as "best" linear approximation. Approximations using differentials. Additional extremal problems. Related computer applications.

7. Intuitive introduction to area. (2 hours) Computational approximation of areas of regions under a curve.

8. Definite integral. (3 hours) Simple quadrature rules and their applications.

9. Indefinite integrals, Fundamental Theorem. (4 hours)

10. Logarithmic and exponential functions. (3 hours) Computer problems involving exponential growth or decay using Euler's method.

11. Applications of integration. (3 hours)

COMMENTARY ON MC-1

1. Introduction. The computer can be used to evaluate functions, thereby considerably extending the kinds of functions a student can handle and, indeed, even recognize as functions (e.g., functions with piecewise or algorithmic definitions can be evaluated numerically even though they may not be expressible algebraically). Students should recognize that the relation between the input and the output of a computer program can define a function; such an awareness can be used later to demonstrate the existence of various interesting functions.

A good computing facility would enable the student to experiment with functions--and also their graphs when they can be drawn--in much the same way as he can work with simple functions when he has only pencil and paper. A graphing program should be provided or developed, and students should become reasonably familiar with it, so that it can be used to motivate later topics in terms of graphs.

2. Limits, continuity. Many kinds of calculations help to motivate the need for a precise definition of limit. Such calculations arise in practical attempts to approximate limits of functions or rates of change. While the student will sense that successive approximations are approaching a limit, he will also discover that the limitations of numerical approximations due to truncation or roundoff errors prevent him from calculating that limit exactly. This awareness should be used to motivate the need for mathematical proofs of the existence of limits.

The bisection method for finding zeros of continuous functions can be introduced either as motivation for or as an application of the Intermediate Value Theorem.

In this section, as well as in others, one should recall several points observed earlier concerning the use or misuse of the computer. First, one should use terminology carefully so as not to mislead students; a computer can approximate a limit, but it cannot "compute" one, nor can it "test" a function for continuity. Second, one should remember that the primary purpose of the course is still to teach calculus and that it is the algorithmic approach, and only secondarily the actual use of the computer, which advances this purpose; hence the computer does not have to be used in every conceivable situation, and many points about algorithms can be made without a computer at all.

3. Differentiation of rational functions. Nonnumerical algorithms can be recognized when they appear even though they may not be programmed. For example, formal differentiation should be recognized as a process that can be mechanized.

More realistic maximum and minimum problems can be attempted. The approach to such problems would include graphing functions, searching for extremal points, and sometimes finding zeros of derivatives. The computer increases the student's power to find zeros of functions since the bisection method or Newton's method are available when algebraic techniques fail.

4. Chain rule. Computer programs and flowcharts can be used to explain the process of functional composition and to motivate the chain rule. Information about the inverse of a function f can be obtained by finding zeros of $f(x) - a$, for various values of a.

5. Differentiation of trigonometric functions. The graphing program can be used for motivation.

6. Applications of differentiation. The limitations of numerical methods can be used to motivate the need for theorems concerning, say, the number of extremal points. For example, numerical methods may lead one to suspect that $x^2 + \cos^2(kx)$ has a unique minimum when k is slightly larger than 1, rather than two minima which are separated by a maximum at 0.

Again, more realistic maximum and minimum problems can be attempted. Newton's method for locating zeros of functions can be developed and compared with the bisection method for its rate of convergence and range of applicability.

7. Intuitive introduction to area.

8. Definite integral. The notion of the definite integral can be made concrete prior to the proof of the Fundamental Theorem so that the student need not confuse the existence of the definite integral of a function with his ability to find an antiderivative. The student can write programs to approximate definite integrals by techniques such as the trapezoidal rule. Improper integrals can be motivated in terms of programs to approximate them.

9. Indefinite integrals, Fundamental Theorem. The nature of the indefinite integral as a function of the upper endpoint can be illustrated by considering a computer program to approximate values of this function.

10. Logarithmic and exponential functions. Numerical methods can be used to discuss and sketch solutions of the differential equation $y' = ky$.

11. Applications of integration. The computer greatly increases the variety of examples which can be treated. Applications of the integral as the limit of Riemann sums, and not merely as an antiderivative, were recommended in the GCMC Commentary and can be handled much more successfully with the use of the computer. For example, in following those suggestions one can use numerical techniques to integrate the normal probability distribution or to graph a logistic curve corresponding to a differential equation $N' = (a - bN)N$ governing population growth. One can also observe the general applicability of numerical techniques as opposed to the often limited applicability of analytical techniques. For example, given experimental data concerning the acceleration of a vehicle, one can compute the values of integrals to obtain the velocity and position of that vehicle [cf. Garfunkel, Solomon. "A laboratory and computer based approach to calculus." American Mathematical Monthly, 79 (1972), pp. 282-290].

COURSE OUTLINE FOR MC-2

1. Techniques of integration. (9 hours) Integration by trigonometric substitutions and by parts; inverse trigonometric functions; quadrature formulas and computer applications; improper

integrals and numerical questions; volumes of solids of revolution.

2. Elementary differential equations. (7 hours) Elementary methods for computational solution.

3. Analytic geometry. (10 hours) Vectors; lines and planes in space; polar coordinates; parametric equations.

4. Partial derivatives. (5 hours)

5. Multiple integrals. (5 hours)

COMMENTARY ON MC-2

1. Techniques of integration. At the discretion of the instructor, less attention might be paid to techniques of formal integration in order to provide time for a study of numerical methods for approximating definite integrals. Experiments can be performed to suggest theorems about the rates of convergence of various methods. In some simple cases one might attempt to place bounds on the numerical errors due to the approximation method and to truncation and roundoff effects. In general, an applied flavor can be introduced into the calculus course by relating some of the theorems to realistic numerical processes.

Although formal integration is more complicated than formal differentiation, certain aspects, such as integration of powers of sines and cosines or the use of partial fractions, can be considered from an algorithmic point of view.

As an example of finding error bounds, consider the midpoint (or tangent) approximation

$$\int_a^b f(x)\,dx \approx h \sum_{k=1}^{n} f(a + [k - \tfrac{1}{2}]h),$$

where $h = \frac{b-a}{n}$, to the integral of a twice-differentiable function f. One can show with the aid of Taylor's theorem that the truncation error is bounded by $\frac{(b-a)h^2}{24}B$, provided that $|f''(x)| \leq B$ for $a < x < b$. Furthermore, for suitable a, b, and h, the error in evaluating the approximation is bounded by $nhE_1 + n(n-1)E_2Fh$, which is less than $(b-a)(E_1 + nE_2F)$, where E_1 is the maximum absolute error in the computation of $f(x)$ for $a < x < b$, E_2 is a bound for $\frac{(1+r)^n - 1}{n}$ (r being the relative roundoff error bound), and $|f(x)| \leqq F$ for $a < x < b$. In the particular case $n = 128$ and $E_1 = E_2 = 1.01 \times 2^{-27}$, the midpoint approximation to $\int_1^2 \frac{dx}{x} = \log 2$ can be guaranteed to have an error of no more than 10^{-5}.

It should be observed that formal and numerical methods are not mutually exclusive alternatives, and that many problems require a combination of the two. Analytical techniques may be used to transform an integral for numerical methods. For example, the integral

$$\int_{\frac{1}{2}\pi}^{\infty} \frac{\sin x}{x}\, dx$$

is more easily handled numerically if it is first transformed to

$$\frac{4}{\pi^2} - 2 \int_{\frac{1}{2}\pi}^{\infty} \frac{\sin x}{x^3}\, dx,$$

which is obtained by integrating by parts twice.

2. Elementary differential equations. The notion of a tangent field can be used to suggest numerical methods for the approximate solution of first-order differential equations. Higher-order equations can also be treated by translating them into systems of first-order equations which can then be solved numerically.

A bound on the propagated error for a simple method can be derived. With Euler's method applied to $y' = f(x,y)$, $y(x_o) = y_o$, it can be shown that the propagated error is bounded by

$$\frac{R + T}{hL} \exp[L(x_n - x_o)],$$

where R is a bound on the local "roundoff" error

$$y_n^c - y_{n-1}^c - hf(x_{n-1}, y_{n-1}^c),$$

T is a bound on the local "truncation" error

$$y(x_n) - y(x_{n-1}) - hf(x_{n-1}, y(x_{n-1})),$$

h is the step-size, L is a Lipschitz constant, and y_n^c is the computer approximation to $y(x_n)$, $x_n > x_o$. [See Gear, C. William. Numerical Initial Value Problems in Ordinary Differential Equations. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1971.]

3, 4, 5. Analytic geometry, Partial derivatives, Multiple integrals. Due to the lack of numerical methods which are both elementary and practical, the computer itself has less impact on the teaching of multivariable calculus than on single-variable calculus. However, an algorithmic approach can still be used, for example, to stress analogies with single-variable calculus or to introduce formal manipulations. If computing applications are desired, one might discuss hill-climbing techniques for finding maxima or some relatively simple method for approximating the values of double integrals.

REFERENCES

The following texts contain elementary applications of numerical methods to the calculus.

1. Sources of applications

Barrodale, Ian; Ehle, Byron L.; Roberts, F. D. K. Elementary Computer Applications in Science, Engineering, and Business. New York, John Wiley and Sons, Inc., 1971.

Dorn, William S.; Bitter, Gary G.; Hector, David L. Computer Applications for Calculus. Boston, Massachusetts, Prindle, Weber, Inc., 1972.

Hamming, Richard W. Calculus and the Computer Revolution. Boston, Massachusetts, Houghton Mifflin Company, 1968.

Hull, Thomas E. and Day, David D. F. Computers and Problem Solving. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1969. Chapter 9.

2. Some calculus texts having a computational flavor

Flanders, Harley; Korfage, Robert R.; Price, Justin J. Calculus. New York, Academic Press, Inc., 1970.

Henriksen, M. and Lees, M. Single-Variable Calculus. New York, Worth Publishers, Inc., 1970.

Stenberg, Warren, et al. Calculus, A Computer Oriented Presentation, Parts 1 and 2. CRICISAM, Florida State University, 1970.

MC-DM. Discrete Mathematics

[Prerequisite: The material of this course can be taught at various levels of difficulty and sophistication in the undergraduate curriculum. Although no specific college mathematics courses are prerequisite for MC-DM, it is important that the student have ability in manipulating symbols and in using formulas.] Although an obvious goal of this course is to equip the students with some useful mathematical tools, a more important goal is to develop their ability to perceive, formulate, and solve problems that are discrete in nature. This course can be taken prior to, or concurrently with, a course in calculus. Indeed, students who are not mathematics majors might be counseled to take such a course rather than the traditional freshman calculus course. (For example, it can be argued that for social science, behavioral science, and biological science majors, a course in discrete mathematics might be more useful than a course in calculus.) There are several undergraduate-level mathematical courses that can follow this course naturally. A course in Probability and Statistics (see, for example, 4.2 below) or a course in Applied Algebra may well be popular choices. Other courses are Combinatorial Mathematics, Optimization Techniques (such as those proposed by the Panel on Applied Mathematics in the CUPM report Applied Mathematics in the Undergraduate Curriculum), Applied Logic, Graph Theory, and Computational Algorithms. Although computing facilities are not absolutely essential in such a course, they can play a very attractive supporting role. Since there is a strong algorithmic flavor throughout this course, the implementation of some computational algorithms will enhance the understanding and appreciation of the mathematical results. Also, probably in a less significant way, ideas such as graphical representation of discrete functions and solution of difference equations can be illustrated on a computer.

The number of hours specified is intended to indicate the relative emphasis for the various topics. Some instructors will find these time estimates unsuitable and will therefore need to make adjustments for their classes. However, because the material covered in this course is so new and unusual in the undergraduate curriculum, the Panel felt it would be valuable to present a wide variety of ideas for a course in discrete mathematics.

COURSE OUTLINE

1. Elementary set theory. (2 hours) Basic concepts and terminology in set theory. Subsets. Empty set. Intersection, union, symmetric difference, and complementation of sets. Venn diagrams.

2. Permutations and combinations. (4 hours) Permutation and combination of objects. Simple enumeration formulas such as that for the number of ways to select or to arrange r objects from n

objects with or without repetitions. Simple machine tools of combinatorics such as computer algorithms for generating all permutations and all combinations of a set of objects.

3. Discrete functions. (2 hours) Domain and range of a function. One-to-one and onto functions. Pigeonhole principle.

4. Manipulation of discrete functions. (2 hours) Forward and backward differences of a discrete function. Accumulated sum of a discrete function. Sum, product, convolution, and correlation of discrete functions.

5. Generating functions. (4 hours) Generating functions as alternative representations of discrete functions. Operations on discrete functions and the corresponding operations on their generating functions.

6. Difference equations. (5 hours) Linear difference equations with constant coefficients. Homogeneous solution and particular solution. Boundary conditions and undetermined coefficients. Solution of difference equations by the technique of generating functions. Simultaneous difference equations.

7. Relations. (2 hours) Cartesian product of sets. Binary relations. Reflexive, symmetric, transitive relations. Equivalence relations. Partial ordering relations. Union, intersection, and complementation of relations.

8. Graphs. (2 hours) Basic terminology in the theory of graphs. Directed graphs. Linear graphs and multigraphs. Connectedness. Paths. Graphs as representations of binary relations. Graphs as structural models.

9. Trees, circuits, and cut-sets. (4 hours) Mathematical properties of trees. Trees as structural models. Spanning trees. Circuits. Cut-sets.

10. Path problems in graphs. (5 hours) Eulerian path. Hamiltonian path. Existence of Eulerian paths and Hamiltonian paths in graphs. Physical interpretation of these notions. Shortest path algorithms and related problems.

11. Network flow problems. (4 hours) Transportation networks. Maximum-flow minimum-cut theorem. The Ford-Fulkerson algorithm for finding maximum flow.

COMMENTARY

1. Elementary set theory. The theme of this course is "discrete objects and their relationships." Consequently, the language of elementary set theory will be used throughout the course. The discussion should remain at an intuitive level, although it is quite reasonable to mention topics such as Russell's paradox which may lead to a discussion of axiomatic set theory.

2. Permutations and combinations. The discussion can begin with a determination of the number of subsets of a given set, a natural continuation of the material in Section 1. An important lesson to teach the students is that often some seemingly difficult problems may have very simple methods of solution when considered from the correct point of view.

Example: Design an algorithm for generating all r-combinations of n objects with unlimited repetitions.

Example: From all 5-digit numbers a number is selected at random. What is the probability that the number selected has its digits arranged in nondescending order?

[Answer: $\binom{10 + 5 - 1}{5} \quad 10^5$]

3. Discrete functions. The notion of discrete functions is introduced as an association of values (elements in the range) to objects (elements in the domain). There are numerous examples of discrete functions: coloring the faces of a polyhedron, assigning grades to students, classification of documents, etc. Point out the obvious extension to the notion of continuous function. The pigeonhole principle (also known as the shoebox argument) is a powerful technique, although it sounds extremely simple, as the following example illustrates.

Example: The integers 1, 2, 3, ..., 101 are arranged randomly in a sequence. Show that there is either a monotonically increasing subsequence or a monotonically decreasing subsequence of 11 (or more) integers.

[Solution: Let $a_1, a_2, a_3, \ldots, a_{101}$ denote a random arrangement of the integers 1, 2, 3, ..., 101. Let us label each integer a_k with a pair of numbers (i_k, j_k), where i_k is the length of a longest monotonically increasing subsequence that begins at a_k, and j_k is the length of a longest monotonically decreasing subsequence that begins at a_k. Suppose that $1 \leqq i_k \leqq 10$, $1 \leqq j_k \leqq 10$ for $k = 1, 2, \ldots, 101$. According to the pigeonhole principle, there must exist a_m and a_n which are labelled with the same pair of numbers. However, this is an impossibility because $a_m < a_n$ implies that $i_m > i_n$, and $a_m > a_n$ implies that $j_m > j_n$. (We assume that $m < n$.)]

4. Manipulation of discrete functions. The notion of the forward and backward differences of a discrete function corresponds to the notion of the derivative of a continuous function. The notion of the accumulated sum of a discrete function corresponds to the notion of the integral of a continuous function. The convolution $z(n)$ of two discrete functions $x(n)$ and $y(n)$ is defined to be

$$z(n) = \sum_i x(i)\, y(n-i).$$

The crosscorrelation function $w(n)$ of two discrete functions $x(n)$ and $y(n)$ is defined to be

$$w(n) = \sum_i x(i)\, y(i-n).$$

The autocorrelation of a function is the crosscorrelation of the function with itself.

Example: Consider the sequence $A = \{1, 1, 1, -1, 1, 1, -1\}$ as a signal transmitted by a radar transmitter. This signal is bounced back by an object whose distance from the radar is to be measured. (The distance can be determined from the elapsed time between the transmission of the signal and the arrival of the return signal.) To minimize the effect of noise interference, we want to choose a sequence so that the correlation function between the transmitted and the received signals will have a large peak value. Show that A is a good choice. [Answer: The autocorrelation function of the sequence A is $\{-1,0,-1,0,-1,0,7,0,-1,0,-1,0,-1\}$, which has a large peak value.]

5. Generating functions. The concept of the generating function of a discrete function corresponds to the concept of the Laplace transformation or the Fourier transformation of a continuous function. The sum of two discrete functions corresponds to the sum of their generating functions. The convolution of two discrete functions corresponds to the product of their generating functions.

Example: Show that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \dots + \binom{n}{n}^2 = \binom{2n}{n}.$$

Give a combinatorial interpretation (in terms of selection of objects) of this equality. [Answer: When a coin is tossed $2n$ times, there are 2^{2n} sequences of possible outcomes. Both sides of the above equality give the number of sequences of outcomes in which the number of heads occurring in the first n tosses is equal to the number of heads occurring in the last n tosses.]

6. Difference equations. Students will be better prepared for a course in differential equations after they have studied difference equations in this course. Indeed, concepts such as homogeneous solutions and particular solutions carry over directly to differential equations. Solving difference equations by the technique of generating functions corresponds to solving differential equations by the technique of Laplace transformations.

Example: A certain nuclear reaction in a system containing nuclei and high and low energy free particles is described as follows. There are two kinds of events: (i) a high energy particle strikes a nucleus, causing it to emit 3 high energy particles and 1 low energy particle, and is absorbed; (ii) a low energy particle strikes a nucleus, causing it to emit 2 high energy particles and 1 low energy particle, and is absorbed. Every free particle causes an event 1 μsec after it is emitted. If a single high energy particle is injected at time $t = 0$ into a system containing only nuclei, what will the total number of free particles in the system be at time $t = 20$ μsec? [Solution: Let a_n

denote the number of high energy particles and b_n the number of low energy particles in the system at the n^{th} μsecond. We have the simultaneous difference equations: $a_n = 3a_{n-1} + 2b_{n-1}$ and $b_n = a_{n-1} + b_{n-1}$, with the initial conditions $a_o = 1$ and $b_o = 0$. Solving these equations, we obtain $a_n + b_n = \frac{1}{2\sqrt{3}} \{(2 + \sqrt{3})^{n+1} - (2 - \sqrt{3})^{n+1}\}$.]

7. Relations. Structural properties of sets of discrete objects can be described by relations. There are numerous examples of the concept of relations between objects: for instance, the relation "is the father of" is nonreflexive, nonsymmetric, and nontransitive; the relation "is the spouse of" is symmetric; the relation "is divisible by" (between integers) is a partial ordering relation.

Example: Write a computer program to determine all possible assignments of 0's and 1's to the vertices of the partial ordering diagram in Fig. 1 so that a 1 never precedes a 0.

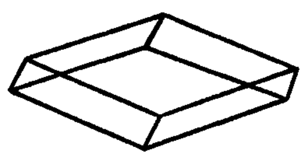


Fig. 1

8. Graphs. There are many examples from various disciplines using graphs as abstract models of structures, among which are social structures, finite state machines, PERT charts, data structures in computer programs.

Example: The inputs to an electronic combination lock are strings of 0's and 1's. The lock will be opened when the pattern 010010 appears at the end of the input string. Such a lock can be modeled graphically as in Fig. 2, where a string of 0's and 1's defines a path starting at the initial vertex.

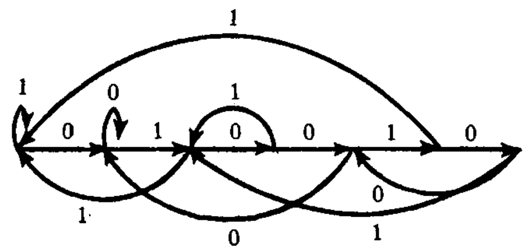


Fig. 2

9. Trees, circuits, and cut-sets. Although trees are very simple in concept, they are rich in structure and find application in many areas of study. There is enormous room for further discussion beyond the basic concepts; topics such as enumeration of trees, optimal trees (notably the Huffman algorithm for determining trees with minimum weighted path lengths), and algorithms for traversing trees may be considered.

Example: Communication links are to be built between cities. Suppose the cost of building a link between two cities is proportional to the distance between them. We want to build a set of links so that there is a path through these links between every two cities. Design a nonexhaustive algorithm that will yield a layout of minimal total cost. (This is a problem of designing an algorithm for finding a minimal spanning tree in a graph with weighted edges.)

10. Path problems in graphs. The notion of a shortest path in a graph has a clear interpretation in physical terms. If computing facilities are available, the implementation of some graph algorithms by students would be highly desirable. The discussion of Eulerian paths brings out another feature in our study of discrete structures--a simple criterion for the existence of some properties in a large class of structures. The following examples also illustrate some physical interpretations of the abstract notions of Eulerian and Hamiltonian paths.

Example: Arrange all n-digit binary numbers in such a way that the last (n-1) digits of a number are equal to the first (n-1) digits of the successive number. (This is an Eulerian path problem with application in digital engineering.)

Example: Arrange all n-digit binary numbers in such a way that two adjacent numbers differ only at one digit. (This is a Hamiltonian path problem with application in digital engineering.)

11. Network flow problems. This discussion not only exposes the students to the general problem of discrete optimization but also shows them a recursive technique in which the solution is improved in a step-by-step manner until an optimal solution is reached. This has exactly the same flavor as that of the simplex method in linear programming.

Example: Engineers and technicians are to be hired by a company to participate in three projects. The personnel requirements of these three projects are listed in the following table:

Minimal number of people needed in each project		Minimal number in each category Mechanical engineers	Mechanical technicians	Electrical engineers	Electrica technicia
Project I	40	5	10	10	5
Project II	40	10	5	15	5
Project III	20	5	0	10	5

Moreover, to prepare for later expansion, the company wants to hire at least 30 mechanical engineers, 20 mechanical technicians, 20 electrical engineers, and 20 electrical technicians. What is a minimal number of persons in each category that the company should hire, and how should they be allocated to the three projects? (This problem can be formulated as a problem of finding a minimal flow in a transportation network where there is a lower bound on the flow-value in each of the edges.)

REFERENCES

The following four books would be useful in a course at the freshman-sophomore level. Berman and Fryer contains a very broad coverage of combinatorics. Kemeny, Snell, and Thompson discusses many interesting application problems. Berztiss was written mainly for computer science students. Vilenkin has a nice collection of examples and problems.

Berman, Gerald and Fryer, K. D. Introduction to Combinatorics. New York, Academic Press, Inc., 1972.

Berztiss, A. T. Data Structures. New York, Academic Press, Inc., 1972.

Kemeny, John G.; Snell, J. Laurie; Thompson, Gerald L. Introduction to Finite Mathematics, 2nd ed. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1966.

Vilenkin, N. Y. Combinatorics. New York, Academic Press, Inc., 1971.

The following books are more suitable for a junior-senior level course. They can also be used as references in a freshman-sophomore level course. Although these books are more advanced than the books cited above, they are quite readable even for undergraduate students.

Berge, Claude. Principles of Combinatorics. New York, Academic Press, Inc., 1971.

Berge, Claude. The Theory of Graphs and its Applications. New York, John Wiley and Sons, Inc., 1962.

Busacker, Robert G. and Saaty, Thomas L. Finite Graphs and Networks: An Introduction with Applications. New York, McGraw-Hill Book Company, 1965.

Knuth, Donald E. The Art of Computer Programming, vol. I. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1968.

Liu, C. L. Introduction to Combinatorial Mathematics. New York, McGraw-Hill Book Company, 1968.

Ryser, Herbert J. Combinatorial Mathematics, MAA Carus Monograph 14. New York, John Wiley and Sons, Inc., 1963.

MC-3. Algorithmic Elementary Linear Algebra.

[Prerequisite: MC-0 or equivalent background] This course corresponds to Mathematics 3, "Elementary Linear Algebra," as described in the GCMC Commentary, and we refer the reader to that report for some additional comments. The differences between the two courses are mainly matters of emphasis and arrangement of topics. Whereas the course described in the GCMC Commentary stresses the algebraic and geometrical aspects of linear algebra and has a certain abstract flavor, the present course has a predominantly algorithmic viewpoint and its discussion revolves around the various ramifications of solving a system of linear equations. Throughout the course, detailed algorithms are to be presented and discussed, in flowchart or some simple step-by-step form, and the students should use these in connection with various practical problems, on a computer where possible. At the same time, in this course it is particularly important to warn the student that the algorithms are based on arithmetic with real numbers and that in a practical computation the effect of roundoff errors may lead to considerable distortions of the final result. This may be illustrated with well-chosen examples, but no attempt should be made to enter into a deeper discussion of such numerical problems.

COURSE OUTLINE

1. Introduction. (3 hours) Discussion of various practical problems involving matrices. Review of the elimination process for 2×2 and 3×3 systems of equations. Examples showing various cases of solvability of such systems.

2. Matrix algebra. (5 hours) Definition of real $n \times m$ matrices. Examples of various special forms of matrices. Transposes; symmetric and diagonal matrices. Equality, addition, and scalar multiplication. Matrix product and its properties. Computational applications of matrix algebra.

3. Vectors and geometry. (4 hours) Geometrical interpretation of 1×3 matrices. Algebraic properties in 3-space. The inner product. Euclidean length, angle, orthogonality, direction cosines. Linear combinations. Lines and planes. Projections. Vector proofs of simple geometric theorems. Matrices as linear transformations in R^2 and R^3. Geometric interpretation of one linear equation in three variables and of a 3×3 system.

4. Inverses and the row echelon form. (5 hours) Left and right inverses of an $n \times m$ matrix and relation to existence and

uniqueness of solutions of linear equations. Review of the elimination process as motivation for the elementary row operations. Elementary row operations and their formulation in terms of multiplication by elementary matrices. The algorithm for transforming a matrix to row echelon form. Equivalent systems of equations. Solvability properties of homogeneous and inhomogeneous systems using row echelon form.

5. Linear dependence and independence. (5 hours) Linear combinations of $n \times m$ matrices. Linear spaces of vectors and matrices. Subspaces. Linear dependence and independence of vectors in R^3 and in R^n, and of matrices. Examples and basic properties. Use of row echelon form to determine linear dependence or independence in R^n. Bases. Exchange algorithm. Dimension. Sum and intersection of subspaces and their dimensions.

6. Elimination. (5 hours) Algorithm for solving triangular systems. Inverses of triangular matrices. Gaussian elimination without pivoting; triangular decomposition. Pivots and the general algorithm. Backsubstitution and the solution of square linear systems. Algorithm for the computation of inverses. Numerical examples of ill-conditioning.

7. Rank. (5 hours) Linear mappings between linear spaces. Range space and null space. Relation between algebra of mappings and of matrices. Uniqueness aspect of row echelon form. Rank of a matrix. Rank of the transpose. Dimensions of null space and range space and related results.

8. Euclidean spaces. (4 hours) The inner product. Schwarz inequality. Euclidean length in R^n. Orthogonal bases. Gram-Schmidt process. Orthogonal projections. The least squares method.

9. [Optional] Abstract vector spaces. Axiomatic definition of vector space over R. Examples. Linear transformations and their algebra. The matrix of a linear transformation with respect to a given basis. Change of basis.

COMMENTARY

1. Introduction. Practical problems involving matrices abound. They may include the adjacency matrix of a street net, a simple resistive electrical network, a Markov chain example, the method of least squares, etc. [See, e.g., Noble, Ben. Applied Linear Algebra. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1969, Chapter 2.]

2. Matrix algebra. The stress here is on the algorithms of matrix algebra. Matrix multiplication can be motivated by practical examples of inner products leading to the product of a $1 \times n$ matrix by an $n \times 1$ matrix. Then the transformation of variables in linear equations readily provides a motivation of the matrix product. The examples introduced earlier can now be elaborated; for example, connectivity of a street net can be determined by forming powers of the adjacency matrix. A subroutine package for matrix algebra may be very useful for these applications.

3. Vectors and geometry. This section is rather standard. For comments we refer the reader to the GCMC Commentary.

4. Inverses and the row echelon form. In this section a basic algorithm is introduced, namely, the reduction to row echelon form; it will play a central role in the remainder of the course. Various applications are possible--for instance, determining solvability properties of a resistive electrical network.

5. Linear dependence and independence. In discussing the use of the row echelon form to determine linear dependence and independence, it is particularly important to illustrate the numerical problems which might occur when a computer is used. This can be motivated well by simple 2- and 3-dimensional examples. If time permits, the role of the exchange algorithms in linear programming can be illustrated by simple examples. [See, e.g., Stiefel, E. L. Introduction to Numerical Mathematics, translated by W. C. Rheinboldt. New York, Academic Press, Inc., 1963.]

6. Elimination. After a thorough discussion of the overall algorithm, it may be desirable to use a well-written subroutine package for computer assignments involving the solution of linear

systems arising in the practical problems introduced earlier. [See, for example, the routines given in Forsythe, George E. and Moler, C. Computer Solution of Linear Algebraic Systems. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1967.]

7. Rank. For some applications of rank, e.g., to chemical reactions, see Chapter 5 of the book by Ben Noble which was cited above.

8. Euclidean spaces. Again, applications abound. In particular, various problems leading to the use of the least squares method can be discussed.

REFERENCES

1. Matrices and linear algebra

Davis, Philip J. The Mathematics of Matrices. Waltham, Massachusetts, Blaisdell Publishing Company, 1965. A well-written elementary introduction to matrices.

Hohn, Franz E. Elementary Matrix Algebra, 2nd ed. New York, The Macmillan Company, 1964. An introductory text which proceeds in a manner similar to the outline above.

Noble, Ben. Applied Linear Algebra. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1969. This excellent text corresponds in spirit and approach to our outline but contains considerably more material and is, in parts, somewhat more advanced.

2. Numerical aspects

Forsythe, George E. and Moler, C. Computer Solution of Linear Algebraic Systems. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1967. A brief, modern, but more advanced text on the basic numerical aspects of solving systems of linear equations.

Fox, Leslie. Introduction to Numerical Linear Algebra. New York, Oxford University Press, Inc., 1965. This text is a good source of instructive examples of error problems in numerical linear algebra.

4. Further Undergraduate Courses

In this section we discuss a rather heterogeneous group of upper-division undergraduate mathematics courses and areas affected by computing. The given list does not exhaust the possibilities and, even for the areas discussed here, there may well be other ways of incorporating the effect of computing. Clearly, at this level there is considerably more flexibility and there are probably many ways of modifying the approaches we suggest here.

For the courses in this section, the programming prerequisites are, of course, more advanced than for the previous courses; the computational facilities may also need to be more flexible. (See Sections 5.1 and 5.2). Further, the knowledge of computing and applied mathematics required by faculty members teaching these courses differs considerably from course to course. Thus for Ordinary Differential Equations (4.1) and Numerical Calculus (4.4) a knowledge of numerical analysis as well as facility in programming are absolutely essential. For Discrete Probability and Computing (4.2) reasonable programming experience in addition to a knowledge of probability is required. For Algebra Courses Influenced by Computing (4.5) a grounding in the algebraic foundations of computer sciences is needed in addition to the more usual kinds of computer expertise. Finally, for Mathematical Computer Modeling (4.3) a thorough knowledge of the applications involved is essential, of course, in addition to the programming and numerical analysis knowledge required by the selected applications.

4.1 Ordinary Differential Equations (3 semester hours)

[Prerequisites: MC-3, Mathematics 4 from the GCMC Commentary, good programming experience] How ordinary differential equations arise in practice. Separation of variables, integrating factors, variation of parameters, substitution. Equations with constant coefficients. Series solutions. Euler's method and a brief treatment of existence and uniqueness. An explicit Runge-Kutta method; a trapezoidal method for stiff systems. [For a discussion of stiff systems, see Gear, C. William. Numerical Initial Value Problems in Ordinary Differential Equations. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1971.] An introduction to boundary value problems.

The purpose of this course is basically the same as that of a more traditional course on ordinary differential equations, except that greater emphasis should be given to practical methods of solution. The most significant change is the inclusion of several carefully chosen numerical methods.

One numerical method is based on a well-established Runge-Kutta formula and is treated in enough detail to permit the writing of a reasonably effective computer program. This method is adequate for nonstiff problems, and there is no need to make more than brief reference to more complicated methods, such as multi-step methods, for these problems. However, one other method is needed for stiff systems of ordinary differential equations. A method based on the trapezoidal rule is included in this course because it is both simple and adequate. Numerical methods for boundary value problems are also included. A more detailed discussion of other numerical methods should be left to courses in numerical analysis.

This course can be followed by a second semester course covering several more advanced topics and exploiting more thoroughly various numerical methods. Topics for such a second semester may be chosen from among the following:

> Series solutions (including special functions), autonomous systems, Laplace transform, comparison theorem, eigenvalues and eigenfunctions, perturbation theory, asymptotic behavior, numerical methods, Galerkin methods, applications.

COURSE OUTLINE

1. Systems of equations. (2 hours) How ordinary differential equations arise in physical, chemical, biological, and economic problems.

2. Elementary analytic methods. (5 hours) Variables separable, e.g., in $y' = 1 - y^2$. Integrating factors, e.g., in $y' + P(x)y = Q(x)$. Substitution, e.g., in $y' = (ax + by)/(cx + dy)$. Variation of parameters. Solving equations with constant coefficients, e.g., the system $y' = ay + bz$, $z' = cy + dz$, or the higher-order equation $y'' + ay' + by = f(x)$. Introduction to series solutions.

3. Euler's method. (5 hours) Brief treatment of an existence and uniqueness theorem (perhaps without a detailed proof) of the Cauchy-Lipschitz kind, which can also be viewed as a theorem about the convergence of a simple numerical procedure. Bound on propagated error with Euler's method. Numerical examples, including a difficult one such as the Volterra equations that often arise in biological problems, e.g., $y' = 2(y - yz)$, $z' = -z + yz$.

4. More efficient numerical methods. (6 hours) Motivation of explicit Runge-Kutta formulas. A complete numerical method, including a strategy for changing step-size (see flowchart given below). Numerical examples, comparison with Euler's method. Note

the generality of the numerical method for systems of first-order equations: it can be used with nonlinear as well as linear equations; moreover, higher-order equations can be reduced to systems of first-order equations. Brief mention of more complicated multi-step methods.

5. Stiff systems of equations. (6 hours) The inability of standard methods to cope efficiently with stiff systems (e.g., with stable linear systems whose eigenvalues differ in magnitude by large factors). A complete numerical method for stiff systems based on the trapezoidal rule. Numerical examples, e.g., $y' = -101y - 100z$, $z' = y$. Compare Runge-Kutta and trapezoidal methods. Brief mention of other methods for stiff systems.

6. Boundary value problems. (10 hours) Elementary theoretical considerations, including an introduction to eigenvalues and eigenfunctions. Shooting methods. Finite difference methods. Mention of Galerkin methods.

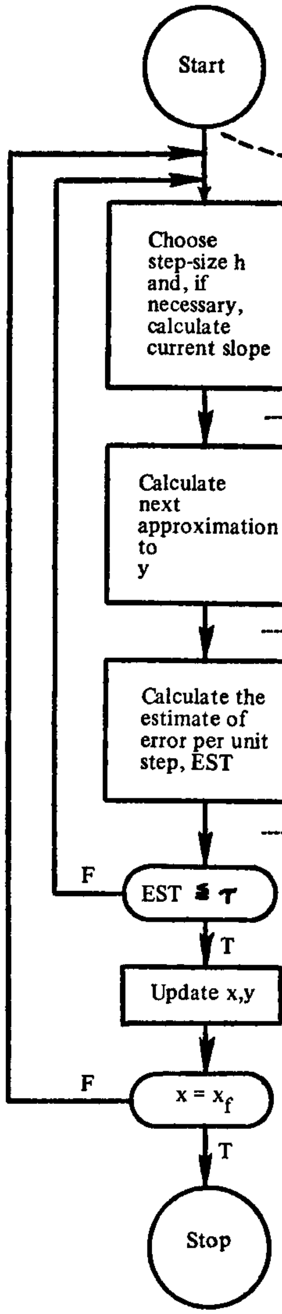
7. Limitations of numerical methods. (2 hours) Acknowledge the limitations of numerical methods and the need for their improvement. Point out the need for further analysis of solutions of differential equations, for example in the neighborhood of a singularity.

COMMENTARY

This course is intended to provide a reasonable balance between analytic and numerical methods that can be applied to problems involving ordinary differential equations. The students are expected to carry out numerical work related to applications.

This theme can be illustrated with Volterra's equations, which are mentioned above in the detailed outline. To begin with, examples of this sort are easily motivated in terms of predator-prey relationships. Then analytic methods can provide some useful information, such as existence and uniqueness of the solutions, and, with certain initial conditions, the existence of periodic solutions. But finding reasonable approximations to the solutions involves the use of numerical methods. The analytic methods are limited to relatively simple problems but help to provide an understanding for more general situations. The numerical methods are much more generally applicable, but

A Runge-Kutta method for nonstiff problems



Start with f (x,y), x_0, y_0, x_f (final value of x), τ (tolerance per unit step), h_{max} (maximum step-size)

$$h = \begin{cases} \min\{h_{max}, x_f - x\}, & \text{on entry} \\ \min\left\{h_{max}, x_f - x, .9\left(\dfrac{\tau}{EST}\right)^{1/4} h_{old}\right\}, & \text{otherwise} \end{cases}$$

Find slope on entry and after successful step

Use Kutta's formula

$[y_{i+1} = y_i + \frac{1}{6}(k_0 + 2k_1 + 2k_2 + k_3)$,

where $k_0 = hf(x_i, y_i)$

$k_1 = hf(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_0)$, etc.] twice with step-size ½h

Find approximation $\bar{y}$ by using Kutta's formula once with step-size h; then EST = $\left\| \dfrac{y - \bar{y}}{15h} \right\|$

Stop with x_0, y_0 replaced by x_f and the computed approximation to $y(x_f)$

they do not contribute very much to one's understanding; moreover, it is often difficult to assess their reliability.

We include in this section a flowchart for the explicit Runge-Kutta method and some comments on a trapezoidal method for stiff systems.

Notes:

1. Choosing h to be $.9(\tau/EST)^{\frac{1}{4}}$ times its previous value can be justified as follows. First of all, the exponent is $\frac{1}{4}$ because the method is a fourth-order method and the ratio $(\tau/EST)^{\frac{1}{4}}$ is asymptotically equal to the ratio of step-sizes associated with errors of τ and EST respectively. The trial step-size should be chosen to be somewhat smaller than what is determined by this ratio, and the factor .9 has been shown experimentally to be reasonably good.

2. Some modifications of the above are needed if we wish to allow $x_f < x_o$.

3. Care should be taken to avoid possible overflow in calculating τ/EST.

4. Provision could be made for using an error exit if the error test fails with h equal to a given h_{min}.

A trapezoidal method for stiff systems

A relatively simple method for stiff systems can be patterned on the flowchart given above. The only major change that needs to be made is to replace Kutta's formula with the trapezoidal formula

$$y_{i+1} = y_i + \frac{h}{2}(y_i' + y_{i+1}')$$

<u>and</u> to arrange for this equation to be solved by Newton's method. (The latter is required because simple iterations on this formula will not usually converge for stiff systems.)

Some minor changes are also needed. The exponent $\frac{1}{4}$ which is used in finding h must be replaced by $\frac{1}{2}$ because the trapezoidal formula is only second-order, and the factor 15 in the formula for EST must be replaced by 3 for the same reason.

REFERENCES

There is no one book which contains all the topics described in this outline. However, the following books taken together cover the material, although the last three especially contain too much for this one course; thus, topics will have to be selected.

Boyce, William E. and DiPrima, Richard C. Elementary Differential Equations and Boundary Value Problems, 2nd ed. New York, John Wiley and Sons, Inc., 1969.

Coddington, Earl A. An Introduction to Ordinary Differential Equations. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1964. Complete coverage of the theoretical aspects.

Davis, Harold T. Introduction to Nonlinear Differential and Integral Equations. New York, Dover Publications, Inc., 1960. Good treatment of practical examples, including the Volterra equations.

Gear, C. William. Numerical Initial Value Problems in Ordinary Differential Equations. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1971. Numerical methods for initial value problems, including stiff systems.

Keller, Herbert B. Numerical Methods for Two-Point Boundary-Value Problems. Boston, Massachusetts, Ginn and Company, 1968.

4.2 Discrete Probability and Computing

[Prerequisites: MC-2 and some knowledge of programming and computing procedures such as those found in Sections 2 and 3 of MC-DM] This course is intended as an introduction to the elements of probability. The main difference between it and a standard probability course, apart from the use of computing, is that, in order to get to more complex problems, less time is spent developing tools for solving simple problems. This difference is reflected in the amount of time allotted to the various units comprising the course, as well as in the fact that difficult theorems (such as the Central Limit Theorem) are to be stated without proof. However, in cases where proofs are omitted, the computer is used to provide experimental intuition for the validity of the theorems.

COURSE OUTLINE

1. Definition of a discrete probability measure; conditional probability for experiments with a finite number of outcomes. (3 hours)

2. The frequency concept of probability; fluctuation theory

illustrated by simulation; the arcsine law for the number of times in the lead. (3 hours)

3. Sums of sequences of independent random variables with common distribution; generating functions; mean; variance. (7 hours) Computational illustration of the Central Limit Thoerem. Proof of the Weak Law of Large Numbers. Illustration of the Strong Law of Large Numbers by simulation.

4. Brief discussion of probabilities on infinite spaces. (4 hours) Poisson, normal, and exponential distributions. Waiting-time problems illustrated by computer simulation.

5. Fair games (martingales). (6 hours) System theorems. Ruin probabilities. The meaning of convergence of nonnegative martingales illustrated using branching processes and other examples.

6. Finite Markov chains. (6 hours) Recurrent and absorbing chains. Use of matrix computation to write programs for basic descriptive quantities relating to Markov chains.

7. Additional topics. (7 hours) Applications of previous topics to selected problems in discrete potential theory, simulation of complex systems, or statistics.

COMMENTARY

1. Definition of a discrete probability measure. This unit represents in part a survey of material from MC-DM. Counting is restricted to permutations and combinations. Computational applications involve the properties of the binomial coefficients.

2. The frequency concept of probability. A possible computer assignment involves the discovery of the highly unintuitive arcsine law for the number of times in the lead in a penny-matching game. Once a conjecture has been established on the basis of experiments, a proof can be given using Feller's treatment based on the reflection principle. This provides an example of an easy limit theorem.

3. Sums of sequences of independent random variables with common distribution. Let $X_1, X_2, \ldots$ be a sequence of independent integer-valued random variables, and let $S_n = X_1 + \ldots + X_n$. A computer program can be used to compute

$$p_j^{(n)} = Pr[S_n = j]$$

using only that

$$p_j^{(n)} = \sum_k p_k^{(1)} \, p_{j-k}^{(n-1)} .$$

This program may then be employed to motivate the concepts of mean and variance and to illustrate the Central Limit Theorem.

4. Brief discussion of probabilities on infinite spaces. This unit is included primarily to provide background for a later course on statistics. The discussion should be limited to distributions for a single experiment, with concepts such as the mean and variance being introduced by analogy with the finite case.

5. Fair games (martingales). Chapter 8 of Kemeny, Schleifer, Snell, and Thompson provides source material. This unit could be replaced by a unit on branching processes and generating functions.

6. Finite Markov chains. For a discussion of some related computational work, see, for example, Kemeny, John G. and Kurtz, Thomas E. Basic Programming, 2nd ed. New York, John Wiley and Sons, Inc., 1971, especially Section 16.3.

7. Additional topics. The purpose of this unit is to unify numerous applications through techniques discussed previously. For example, discrete potential theory can be applied to optimal stopping problems and to Markov decision processes; and the solution of the Dirichlet problem can be found using (a) the voltage in an electrical network, (b) the value of a stopped martingale (the Monte Carlo method), and (c) Markov chain methods. Such applications would build upon units 5 or 6 or both. Alternative or additional applications could include the simulation of complex systems (cf. Forester) or an introduction to elementary statistics.

REFERENCES

There is no single text which is suitable for the entire course. Sections of Feller and of Kemeny, Schleifer, Snell, and Thompson can be used for various units of the mathematical topics, i.e., noncomputational aspects. Freiberger gives a more advanced treatment, and Forester is an example of an application of these ideas to a real-life problem.

1. Mathematical background

Feller, William. An Introduction to Probability Theory and Its Applications, vol. 1, 3rd ed. New York, John Wiley and Sons, Inc., 1968.

Kemeny, John G.; Schleifer, A.; Snell, J. Laurie; Thompson, Gerald L. Finite Mathematics with Business Applications, 2nd ed. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1972.

2. Some computational applications

Forester, Jay W. Urban Dynamics. Cambridge, Massachusetts, MIT Press, 1969.

Freiberger, Walter F. and Grenander, Ulf. A Short Course in Computational Probability and Statistics. New York, Springer-Verlag New York, Inc., 1971.

4.3 Experimental Development of a Course in Mathematical-Computer Modeling

A mathematical model of a phenomenon, mechanism, or process can be a system of algebraic, differential, difference, or functional equations, a stochastic process, or an abstract structure in terms of which a problem or question can be studied and can be given a mathematical solution. The usefulness of mathematical models in the physical sciences and engineering is beyond question; in many instances the models are so good that computer simulation is as accurate as any experimental measurements that can be made. The power of the computer to simulate and to compute widens the scope of acceptable models, affects the usefulness of mathematical methods, and makes possible procedures which are much different from those of the past and far superior to them.

In view of the complexity of physical phenomena which have been successfully subjected to mathematical analysis, mathematicians and scientists do not doubt that useful mathematical models can be constructed in all of the sciences. Indeed, for a long time we have witnessed a growing mathematization within the nonphysical sciences.

In all of this we are just beginning to appreciate the impact of the computer, and we are even less aware of the impact which computing and the computer will eventually have upon mathematics and pedagogy. Today our mathematical instruction is barely beyond the pencil-and-paper and chalk-and-blackboard stage; relatively few mathematicians have had experience in mathematical modeling and in effective use of the computer.

Although it seems imperative today to re-examine the content of our courses and to give our students some training in the processes by which mathematics is and can be applied, it is certainly beyond our experience at the moment to do so extensively at an elementary level. Modeling itself might best be introduced as an integral part of courses designed to teach a certain body of mathematics, but initially it might be better and easier to gain experience by experimenting with a separate course in mathematical-computer modeling at a post-calculus level. This could be a joint experimental undertaking by a number of faculty members and a few students; it should consist of the study and investigation in depth of a small number of carefully selected problems.

In selecting a problem one should take the following things into account:

(1) The problem should be easily stated. Without requiring extensive specialized knowledge or background, it should be possible to distinguish enough of the essential features in order to begin to construct some mathematical models, however crude.

(2) The problem should have mathematical content--the simpler the better at this level--which illustrates how mathematics is needed (i) to provide insight, (ii) to test the model (e.g., against a simple special case where the solution is obvious or easy), (iii) to develop a theory of the essential features of the model, and (iv) to indicate computational procedures.

(3) The problem should in some essential way require use of the computer (i) to provide insight through computer experimentation with the model or problem, (ii) to provide approximate answers and practical solutions, and (iii) to test the model and the solutions.

This does not imply that it is impossible to learn a great deal about modeling with pencil and paper, but a basic objective here is to go beyond this stage and to learn something about the uses and misuses of the computer and mathematical theory. More time needs to be spent in thinking about what goes into and what comes out of the computer than about the computation itself.

It is within the rules of the game to use mathematical or scientific results without proof, although where proofs are easily accessible and instructive they could be included. It would also be good pedagogy to consider models which are known to be poor, impractical theories and solutions, and poor numerical methods.

A SAMPLE PROBLEM

An excellent example is suggested by the work of Harold W. Kuhn. See his papers listed in the references at the end of this section; see also Courant and Robbins.

Fermat-Steiner-Weber Problem

Given n distinct points $p^1 = (x_1,y_1)$, $p^2 = (x_2,y_2)$, ..., $p^n = (x_n,y_n)$ in the plane and n positive numbers $w_1, w_2, \ldots, w_n$, find points which minimize

$$F(p) = \sum_{i=1}^{n} w_i |p - p^i| ,$$

where $p = (x,y)$ and $|p| = (x^2 + y^2)^{\frac{1}{2}}$ (thus $|p - p^i|$ is the Euclidean distance between p and p^i). This problem, posed by Fermat in the early 17th century with $n = 3$ and $w_1 = w_2 = w_3 = 1$, has had a long history and has been studied recently with renewed interest because of applications to spatial economics (optimal location of a factory, a shopping center, a hospital, a communications center, etc.).

Omitting the trivial case when the n points are collinear, we can show without difficulty that F is strictly convex, has a unique minimum which is in the convex hull of $p^1, p^2, \ldots, p^n$, and that the vanishing of a gradient (suitably defined at the vertices p^j) is a necessary and sufficient condition for a minimum.

The history and theory is interesting and provides a necessary background to the problem of finding approximate solutions numerically. The computational difficulties are nontrivial.

The following algorithm has been independently proposed at least three times:

Let

$$q^n = T(q^{n-1}),$$

where

$$T(q) = q + h(q)\ \nabla F(q),$$

$$h(q) = \left(\sum_{k=1}^{n} w_k\ |q - p^k|^{-1}\right)^{-1}$$

with $T(p^k) = p^k$ at the vertices [$h(q)$ is the harmonic mean of the distances to the vertices].

It can be shown that:

If q minimizes F, then it is a fixed point of T. If q

is a fixed point of T that is not a vertex, then q minimizes F. Either (1) $T^n(q)$ converges to a fixed point or (2) $T^j(q) = p^k$ for some j and some k.

Kuhn gives an algorithm which controls the step-size $h(q)\ \nabla F(q)$ for which he conjectures that $F(q^{n+1}) \leqq F(q^n)$. This would imply convergence. Calculations involving $n = 3$ to $n = 24$ give close approximations after seven iterations.

Outline for the Study of this Problem

1. Nonmathematical statement and discussion of an economic problem involving the optimal location of a plant, shopping center, etc.

2. Mathematical statement of the problem. Locate in the plane a point that minimizes the weighted sum of its distances to n given points in the plane.

3. History of the problem. Solution of simple cases. Simplest case (3 points, equal weights) considered by Fermat (c. 1635) in an essay on maximum and minimum problems. The more general problem with weights w_1, w_2, w_3 appears in an early book on "fluxions" by Simpson, one of the first textbooks on calculus.

4. Some mathematical theory.
   a. Existence-uniqueness.
   b. Necessary and sufficient conditions.
   c. Dual problem.

5. Computational methods. Use of the computer.
   a. As a problem in mathematical programming.
   b. A proposed algorithm and its motivation. Iterations, convergence, and fixed points.
   c. Computation of some examples.
   d. The conjecture $F(q^{n+1}) \leqq F(q^n)$. Special cases in which it can be verified.
   e. Computer tests of the conjecture.

6. A specific application. Study the problem of a good location for a large regional high school in the community.

7. Generalizations and unanswered mathematical and practical questions (research problems).

Desirable Features Illustrated by the Example

1. It is simple to describe, easily understood, explicit, interesting, and significant.

2. It has deep roots within the history of mathematics. Special cases of this problem appear as exercises in the earliest texts on "fluxions." It can be considered today in the light of new ideas, new mathematics, and computational procedures related to modern digital computers.

3. It serves to review and illustrate mathematics to which the student has been exposed: max-min, Lagrange multipliers (not required if the dual problem is omitted), simple linear algebra (analytic geometry), convergence.

4. It requires introduction at an elementary level of some new mathematics and new ideas important in mathematics and applications: convexity, duality, iteration (successive approximations), fixed points, and mathematical programming.

5. It provides an opportunity to develop a small body of theory.

6. It raises questions of computation, significant examples of which require the computer.

7. It raises a conjecture which can be proved in special cases and can be tested on the computer in more general cases.

8. It reaches the frontiers of research (generalizations to nonlinear costs, noneuclidean distance, etc., which are significant for applications).

A RECOMMENDATION

The development of individual topics, problems, exercises, etc., needed for a course of this type will require considerable work and imagination. This might be accomplished through isolated projects for independent group study with selected students, directed by an applications- and computer-oriented mathematician and a colleague representing the area of application.

Such experimental courses would be the testing ground for the development of instructional model building and are encouraged by the Panel. In the long run we believe that such model building should come in directly as a vehicle for teaching mathematics and its applications (for an example see the book by Grenander and the book by Freiberger and Grenander).

REFERENCES

Courant, Richard and Robbins, Herbert. What is Mathematics? New York, Oxford University Press, Inc., 1941, p. 354 ff.

Freiberger, Walter F. and Grenander, Ulf. A Short Course in Computational Probability and Statistics. New York, Springer-Verlag New York, Inc., 1971.

Grenander, Ulf. Computational Probability No. 10: Computational Probability and Statistics. Providence, Rhode Island, Brown University, 1971.

Kuhn, Harold W. "Locational problems and mathematical programming." Mathematical Optimization in Economics. Centro Internazionale Matematico Estivo, III Ciclo, L'Aquile, 29 Agosto-7 Settembre 1965. Roma, Edizioni Cremonese, 1966.

Kuhn, Harold W. "On a pair of dual nonlinear programs." Methods of Nonlinear Programming. Amsterdam, North-Holland Publishing Company, 1967, pp. 39-54.

Kuhn, Harold W. and Kuenne, Robert E. "An efficient algorithm for the numerical solution of the generalized Weber problem in spatial economics." Journal of Regional Science, 4 (1962), pp. 21-33.

SUPPLEMENTARY REFERENCES

The following books are meant to illustrate some areas and sources of ideas for modeling. Do not expect to find completely worked out instructional material.

Arrow, Kenneth J., ed. Selected Readings in Economic Theory from Econometrica. Cambridge, Massachusetts, MIT Press, 1971.

Athans, Michael A. and Falb, Peter. Optimal Controls. New York, McGraw-Hill Book Company, 1966.

Bellman, Richard. Introduction to the Mathematical Theory of Control Processes, vol. I. (Linear Equations and Quadratic Criteria). New York, Academic Press, Inc., 1967.

Canon, M.; Cullum, C.; Polak, E. Theory of Optimal Control and Mathematical Programming. New York, McGraw-Hill Book Company, 1970.

Computers in Undergraduate Science Education. Conference Proceedings, Chicago, Illinois, August 17-21, 1970. College Park, Maryland, Commission on College Physics, 1971.

Forester, Jay W. Urban Dynamics. Cambridge, Massachusetts, MIT Press, 1969.

Friedrich, O. "Population explosion: Is man really doomed?" Time, September 13, 1971, pp. 58-59.

Gerstenhaber, Murray, et al. AMS Lectures on Mathematics in the Life Sciences: Some Mathematical Problems in Biology, vol. I. Providence, Rhode Island, American Mathematical Society, 1968.

Hooper, John W. and Nerlove, Marc, eds. Selected Readings in Econometrics from Econometrica. Cambridge, Massachusetts, MIT Press, 1970.

IEEE Conference on Decision and Control. Institute of Electrical and Electronic Engineers, 345 East 47th Street, New York, New York 10017, 1971.

Joint Automatic Control Conference of the American Automatic Control Council. Institute of Electrical and Electronic Engineers, 345 East 47th Street, New York, New York 10017, 1971.

Kirk, Donald E. Optimal Control Theory, An Introduction. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1970.

Koenig, H. E., et al. Analysis of Discrete Physical Systems. New York, McGraw-Hill Book Company, 1967.

Ledley, Robert S. Use of Computers in Biology and Medicine. New York, McGraw-Hill Book Company, 1965.

Meyerson, Martin and Banfield, Edward C. Politics, Planning, and the Public Interest. New York, Free Press, 1955.

Noble, Ben. Applications of Undergraduate Mathematics in Engineering. New York, The Macmillan Company, 1967.

Pielou, E. C. An Introduction to Mathematical Ecology. New York, John Wiley and Sons, Inc., 1969.

Polak, E. Computational Methods in Optimization. New York, Academic Press, Inc., 1971.

Shapiro, George, et al. Prospects for Simulation and Simulators of Dynamic Systems. New York, Spartan Books, Inc., 1967.

Solow, Robert M. "The economist's approach to pollution and its control." Science, 173 (1971), pp. 498-503.

Steinitz, Carl and Rogers, Peter. Systems Analysis Model of Urbanization and Change. Cambridge, Massachusetts, MIT Press, 1970.

Sternberg, S. "Stochastic Learning Theory." Handbook of Mathematical Psychology, vol. II. New York, John Wiley and Sons, Inc., 1963.

Wagner, Harvey M. Principles of Operations Research with Applications to Managerial Decisions. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1969.

Wodwell, G.; Craig, P.; Johnson, H. A. "DDT in the biosphere: Where will it go?" Science, 174 (1971), pp. 1101-1107.

Zadeh, Lotfi and Polak, Elijah. System Theory. New York, McGraw-Hill Book Company, 1969.

4.4 Thoughts on a "Postponed" Calculus Course with Emphasis on Numerical Methods

In recent years many questions have been raised about the special role played by the basic calculus sequence as the first set of courses in traditional college mathematics curricula. There are many arguments for beginning with the calculus, but with the growth of computer science and the need for more mathematics in the behavioral and social sciences there are more and more arguments for postponing the calculus courses.

For those students who do not need to use the calculus in other courses until the junior or senior year, a drastically revised one-year calculus course which makes heavy use of computing and algorithmic ideas may be suitable. This course would have the Discrete Mathematics course MC-DM and a thorough knowledge of programming as prerequisites and would not be taken until the sophomore or junior year. A constructive approach to the basic concepts of the calculus would be used throughout the course and heavy emphasis would be placed on both numerical and nonnumerical algorithms. The course would contain some elementary numerical analysis, attention being paid to error analysis and degrees of approximation.

By necessity, some of the traditional topics of the calculus will have to be slighted, but the knowledge that the students will gain in being able to handle fairly complex real-world problems would certainly offset this.

The following outline should be considered as a first tentative suggestion. Given the novelty of the approach, there are very few experiences which might have been used as a guide. The material is ample for a one-year course, but no attempt is made to indicate the pace. The increased mathematical maturity of the students should make possible a faster pace than that in the usual calculus course. It should be kept in mind throughout the course that the topics are to be treated with heavy emphasis on numerical orientation.

A TENTATIVE OUTLINE

1. Numbers. A brief review of (intuitive) number concepts. Distribution of floating-point numbers on the line. Arithmetic problems with floating-point numbers. Roundoff errors. Ordering, inequalities, distances, and absolute value. All of this should be computationally oriented.

2. Sequences. Computational example of approximating the square root. Squeeze concept. Other related examples of limits. Need for irrational numbers to "fill" the number line; completeness concept. Definition of limit. Basic limit theorems (prove only a few). Squeeze theorem. Importance of error estimates. Slow and fast convergence illustrated by various examples.

3. Functions. Review of function concept (functions as mappings). Functions defined by algorithms, e.g., Horner's scheme, etc. Graphs and basic curve sketching. Arithmetic combination of functions. Geometric discussion of "near" functions and simple computational examples of approximations. Composition of functions, inverses. Monotonicity. Zeros, bisection algorithm. Uniform continuity; Intermediate Value Theorem for uniformly continuous functions using bisection algorithm.

4. Interpolation. Polynomial interpolation, undetermined coefficients, Lagrange formula, application to the solution of equations.

5. Derivatives. Limits, basic limit theorems with reference to the sequential case. Motivation and definition of the derivative. The cases x^k (small k) and $1/x$. Concept of higher derivatives. Continuity. Basic differentiation theorems. Derivative of polynomials, Horner's algorithm again. Derivative of rational functions. Linearization. Newton's method. Monotone convergence of Newton's method. Derivatives of monotone, convex, and concave functions. Chain rule. Implicit functions, inverse functions, application to $x^{1/n}$.

6. Area. Intuitive discussion of properties of area. Area of regions under monotone functions by approximations with sums of rectangles. Extension to nonmonotone functions, application to x^k, $k = 0, 1, 2, 3$.

7. Integral. Riemann sums, existence for uniformly continuous functions, basic properties. The Fundamental Theorem of Calculus. Application to the calculation of definite integrals. Substitution and integration by parts.

8. Quadrature. Review of rectangular approximation, trapezoidal rule and Simpson's rule. Integration by the Lagrange formula. Algorithmic treatment of partial fractions.

9. Differential equations. Direction fields, concept of solving first-order initial value problems. Separable case. Euler's method. Differential equations of radioactive decay, first-order logarithms. Second-order linear equations, superposition principle, harmonic motion, trigonometric functions.

10. Taylor's theorem. Mean Value Theorem, Taylor's theorem. Lagrange and integral remainder, application to error of interpolation, quadrature, l'Hôpital's rule, critical points, simple numerical methods for critical points.

11. Numerical solution of differential equations. Euler's method reviewed, trapezoidal rule, local discretization error, modified Euler's methods, Taylor's polynomial methods, idea of Runge-Kutta and multi-step methods. Brief geometric discussion of stability problems.

REFERENCES

There is no single textbook which covers the material proposed here, but parts of the following three texts may be used.

Flanders, Harley; Korfhage, Robert R.; Price, Justin J. Calculus. New York, Academic Press, Inc., 1970.

Henriksen, M. and Lees, M. Single-Variable Calculus. New York, Worth Publishers, Inc., 1970.

Stenberg, Warren, et al. Calculus, A Computer Oriented Presentation, Parts 1 and 2. CRICISAM, Florida State University, 1970.

For the numerical analysis portions, parts of various standard books on numerical methods can be used, especially for problems and applications.

4.5 Algebra Courses Influenced by Computing

At the present time it is not clear how the standard undergraduate introduction to algebra (e.g., Mathematics 6M in the GCMC Commentary) should be modified to reflect the growing influence of computers. Some knowledge of algebra is essential for an understanding of areas such as algebraic algorithms and symbol manipulation which have a strong algebraic flavor. Nonetheless, there is no consensus as to how the usual introduction to abstract algebra should be modified. In the discussion below we present brief outlines of three possible modifications, along with some sources of further information.

1. At Harvard University the Department of Applied Mathematics has taught a one-year course based on Birkhoff and Bartee, Modern Applied Algebra (New York, McGraw-Hill Book Company, 1970). Topics are selected from among the following:

> Sets and functions, relations, graphs. Finite state machines, programming languages. Monoids, groups, lattices, Boolean algebras, rings, polynomials, finite fields. Optimization and computer design, binary group codes, polynomial codes, recurrent sequences, computability.

For further details about the course, the book by Birkhoff and Bartee should be consulted.

2. Professor John Lipson of the University of Toronto has taught a modification of the one-year algebra course to advanced students in computer science for the past three years. Lecture notes for this course are expected to be available to interested parties sometime in 1973.

The principal topic in the second half of this course is a study of algebraic algorithms which incorporates recent work not readily available in the textbook literature. The following topics are considered:

> Sets, relations, functions. Examples of algebraic systems. Universal algebra. Lattices, Boolean algebra, groups, rings, finite fields. Interpolation theory, algebraic algorithms.

In addition to the usual textbooks in algebra, the following sources are used:

Berztiss, A. T. Data Structures. New York, Academic Press, Inc., 1971.

Birkhoff, Garrett. Lattice Theory, 3rd ed. Providence, Rhode Island, American Mathematical Society, 1966.

Birkhoff, Garrett and Bartee, Thomas C. Modern Applied Algebra. New York, McGraw-Hill Book Company, 1970.

Knuth, Donald E. The Art of Computer Programming. Reading, Massachusetts, Addison-Wesley Publishing Company. Vol. I, 1968; Vol. II, 1969.

3. A one-semester modification of the course described has been taught in the Department of Electrical Engineering at the Massachusetts Institute of Technology and in the Division of Applied Mathematics at Brown University. The following topics are covered:

> Sets, relations, functions, morphisms, diagram graphs and applications. Monoids, groups, lattices. Finite state machines, semantics of flow diagrams, programming languages. Rings, fields, polynomials, extension fields, finite fields.

5. Implementation

5.1 Computing Facilities

See Section 3.2 of Recommendations for an Undergraduate Program in Computational Mathematics, page 547.

5.2 Programming Requirements

The principal objective of any of the courses described in this report is to describe a mathematical subject area and applications related to it. Thus, the teaching of programming should not, by itself, be a purpose of any of these courses. Ideally, a student entering any of the lower-division courses except MC-0 should be required to have at least a beginning knowledge of one of the standard algorithmic languages implemented at his institution, as well as the ability to develop flowcharts and basic programs from a general description of a process. For the upper-division courses a more thorough familiarity with such a language and more programming expertise is required.

At present, few students entering the lower-division courses will have the corresponding programming background, although the expanding use of computers in high schools may change this picture in the future. Meanwhile, there are several alternatives that can be adopted.

If a student's schedule permits, one solution would be for him to take a one-semester introduction to computing, such as the course C1 described in the CUPM report Recommendations for an Undergraduate Program in Computational Mathematics. If this approach leads to delays in the mathematical progress, a possible alternative might be to let him take the computing course and his first mathematics course concurrently. In that case any one of the courses in Section 3 could be modified and taught in such a way that programming is not absolutely essential, although the results of the computations and the problems raised by computations would, of course, be used in the course.

Another alternative not involving a separate computing course is to teach a minimum amount of programming in supplementary lectures to those who need it during the first few weeks of the freshman courses. The time required for this depends considerably on the computing facilities available and on the language used; here computer use in a conversational mode is often particularly helpful. It is essential that the students be given ample opportunity to write and run programs of their own and to operate the necessary equipment, such as terminals or key punches. Moreover, it is important that consultants be available who can help them over their difficulties without overwhelming them with technical details. In courses where additional credit is given for the computational work, the supplementary programming lectures would, of course, take up the first few of the laboratory sessions held throughout the semester.

Which of these alternatives is the most feasible in a given situation depends not only upon the intended use of the computer in the course but also upon the nature of the available computing facilities.

As mentioned before, a few lectures in programming are not sufficient preparation for the more advanced courses. A consistent programming experience in the lower courses may, in general, enable a student to read an introductory computer science text on his own and to round out his computer knowledge in this way. Otherwise, a first computing course such as the course C1 cited earlier is certainly a natural prerequisite for the upper-division courses.

5.3 Changes in Instructional Techniques

In connection with the general topic of this report it is appropriate to review the state of teaching techniques in light of requirements for incorporating computers into the curriculum and to develop new teaching methods for bringing computational results and numerical algorithms into the classroom. The principal objective is to foster the "laboratory" atmosphere in class and to make each student feel that he is actively engaged in learning through problem solving, experimentation, and discovery.

It is important to bring the computational results into the classroom. Although thoughtful students will learn well from programming projects assigned as homework, the hurried or less thoughtful students see these assignments as chores to be done as quickly as possible. Sometimes a student will turn in a program with an error in it so gross as to make his answers meaningless. He will not have learned anything from the activity unless the instructor is able to review the work in class and exhibit the results which the problem was intended to elucidate.

The college mathematics teacher has always been at a real disadvantage when asked to make his lectures with chalk and blackboard as exciting and interesting as those of, say, his colleagues in physics who have carefully orchestrated, and often dramatic, experiments to perform in class. More than ever, though, we find chalk and blackboard inadequate for the presentation of the new material being proposed in this report; a teacher filling a board with computer results to six significant digits is likely to deter even the most energetic student! We hope that authors and publishers will address themselves to this problem and begin to develop new teaching materials for the mathematics teacher. Three possibilities are mentioned below, in order of increasing cost and complexity.

The first and most accessible teaching assist might come from sets of transparencies to be used with an overhead projector. Graphs of functions of one and two dimensions, successively "blown-up" portions of them, and computational results can all be presented. Carefully prepared overlays can give graphical results a dynamic sense. We are all familiar with the power and appeal of really good, professionally executed illustrations in textbooks. A library of transparencies of equal excellence with which a teacher could illustrate his lecture would go far toward livening up the classroom. The teacher interested in developing visual material should seek help from a media specialist.[1]

The second possibility to be considered is that of videotaped or filmed presentations. Here the dynamic nature of the algorithms can be well conveyed. For illustration, let us consider the concept of the definite integral. If the limit definition is phrased in an algorithmic form, the student will comprehend it best if he sees the approximating rectangles sketched, their areas added in one at a time, and the whole process repeated for a finer partition. When the partition is refined, he sees the effect of taking a larger number of smaller contributions to the integral. It is very difficult to draw accurately enough and fast enough on a blackboard to give students this sense of dynamism. Also, when animation is under consideration, it is natural to try to incorporate computer-produced graphics in these presentations.[2]

---

1. Some information might also be obtained from the Association for Educational Communications and Technology, 1201 16th Street, N.W., Washington, D. C. 20036.

2. Advice may be obtained from Educational Development Center, 55 Chapel Street, Newton, Massachusetts 02160.

An independent reason for developing recorded presentations is that television cassette technology is reaching a stage which will allow a student to view a presentation independently, making individualized instruction a reality. A "library" of cassettes will make it possible for him to spend as much time as necessary on precisely the material that is appropriate for him. Courses could become modular in nature and it would no longer be necessary for all students to proceed in lock-step through the material. We note that freshman classes are becoming increasingly heterogeneous, both with respect to the students' capabilities and to the quality and quantity of their high school mathematics preparation. As "learning centers" with carrels containing TV screens and other audio-visual devices become increasingly common, the mathematical community should be concerned with their potential impact and usefulness.

We encourage authors who wish to prepare materials utilizing these new media to seek professional help from audio-visual specialists. Television and film offer new opportunities for innovative teaching. Simply to televise or to film traditional lectures would fail to take full advantage of the possibilities afforded by these media.

The third and most sophisticated and desirable technological solution is to have an on-line terminal connected to a reliable computer available at all times in the classroom. Devices are available which tap the input to a cathode ray tube display device and put the same image on one (or more) television monitors so that a large class can "participate" in the interaction.[3] If an on-line computer is used, a great deal of preliminary work is required on the part of the teacher. Numerical experiments must be chosen with great care, lest roundoff errors, the peculiarities of the computer operating system, etc., produce unanticipated results. Thus "inverting" a nearly singular matrix or "summing" an alternating series with terms alike to 6 digits using 5-digit arithmetic would obscure rather than illuminate, and could carry the teacher far deeper into the theory of computation than he ever intended to go. These problems are particularly likely to arise if a mini-computer with a small word length is used with only single precision arithmetic.

3. For an overview of these technological developments, we recommend Ronald Blum, ed., Computers in Undergraduate Science Education Conference Proceedings, Commission on College Physics, College Park, Maryland, 1971 (available from American Institute of Physics, 335 East 45th Street, New York, New York 10017). See also Proceedings of a Conference on Computers in the Undergraduate Curricula, 1970 (available from the University of Iowa Computer Center, Iowa City, Iowa 52240), Proceedings of the Second Annual Conference on Computers in the Undergraduate Curricula, 1971 (available from The New England Press, Box 979, Hanover, New Hampshire 03755), and Proceedings of the 1972 Conference on Computers in the Undergraduate Curricula, 1972 (available from Southern Regional Education Board, 130 Sixth Street, N.W., Atlanta, Georgia 30313).