RECOMMENDATIONS ON COURSE CONTENT

FOR THE TRAINING OF TEACHERS OF MATHEMATICS

A Report of

The Panel on Teacher Training

August 1971

TABLE OF CONTENTS

THE PURPOSE OF THIS REPORT

This report presents an outline of the Panel's recommendations for the minimal college preparation for teachers of school mathematics based upon its assessment of those significant changes that have taken place or can be expected to take place in school curricula during the 1970's.

The nature of school mathematics is, of course, far from static, and the forces for change are many. The past 25 years have produced a phonomenal increase in the quantity of known mathematics, as well as in the variety and depth of its applications. This growth has been reflected in our total culture, which has become increasingly mathematical, a trend which is certain to continue. It is inevitable and proper that these changes will be reflected in the content of school mathematics, as well as in the way it is taught.

Thus, in the past ten years we have seen a flurry of activity directed toward improving the mathematics curricula in our schools. The pace of change alone demands that those engaged in such activity periodically review their efforts. A decade seems to be an appropriate period for such a review.

It is reasonable to ask what specific changes in mathematics and mathematics education during the past decade impel us to modify our previous recommendations.

The dependence of western civilization on technology has long been evident, and it has been recognized that mathematics supports the physical and engineering sciences upon which technology thrives. More recently new applications of mathematics to the biological, environmental, and social sciences have developed. Statistics and probability have emerged as important tools in these applications. Indeed, as ordinary citizens we frequently encounter surveys and predictions that make use of probability and statistics, so that an intelligent existence demands our understanding of statistical methods. Consequently, many secondary schools have begun to teach probability and statistics and, as we shall show, there are compelling reasons to teach these subjects in the elementary grades.

But aside from the specific mathematics, e.g., statistics, which is brought to bear upon applications, the applications are interesting in themselves, so that it is pedagogically sound to incorporate them in the mathematics program. Thus, our new recommendations emphasize the applications of mathematics.

Many of these new applications have been aided, perhaps even made possible, by modern computers, and the teaching of computer science and computational mathematics are becoming commonplace in our colleges. Computer programming is now a part of many junior and senior high school mathematics programs, and it has been discovered that the notion of a flowchart for describing an algorithmic process

is a useful pedagogical device in the teaching of elementary mathematics, as well as an efficient device for prescribing a computer program. Thus, computers are influencing mathematics education at all levels, and we have attempted in preparing this report to assess this influence and to recommend measures for increasing it.

It is a fact that change induces change. For instance, an important aspect of curricular change over the past decade has been emphasis on the understanding of mathematical concepts. As a result, we have learned that abstract concepts can be assimilated at a much earlier age than was previously thought possible. Thus, we are less reluctant today to suggest that elementary notions of probability may be useful in explaining ideas about sets and rational numbers than we were a decade ago to suggest that elementary ideas about sets might be useful in helping children to understand the process of counting. Furthermore, the curricular revisions of the past decade have led to improved training programs which have produced elementary school teachers who are more confident about presenting mathematical topics. It is our purpose in this report to take advantage of today's teachers' new attitudes and skills in order to meet new challenges.

Finally, our recommendations are intended to reflect improved preparation over the past decade of entering college freshmean.

THE OBJECTIVES OF TEACHER TRAINING

The Panel believes that the following objectives of mathematical training are important:

1. Understanding of the concepts, structure, and style of mathematics

2. Facility with its applications

3. Ability to solve mathematical problems

4. Development of computational skills

These statements deserve amplification.

It is our belief that the disciplined, rational man has the best chance of becoming independent, mature, and creative, and that the development of these qualities is a lifelong process. The intellectual discipline of mathematics contributes in a unique way to this development. We identify two reasons why this is so. First, mathematical concepts are necessarily rooted in man's awareness of the physical world. Understanding mathematics allows him to relate more efficiently to his environment. Second, a person's understanding of

a concept depends upon its meaningful relation to and firm grounding in his personal experience, as well as upon his awareness of its role in a system of interrelated ideas. As he learns to relate concepts to one another in an orderly fashion he becomes better organized and he improves his ability to abstract and to generalize, that is, to recognize a concept in a variety of specific examples and to apply this concept in differing contexts. We believe that the study of mathematics can directly benefit this process of personal organization. We therefore regard it as essential that mathematics be taught at all levels in such a way as to emphasize its concepts, structure, and style.

It is possible, of course, to study, to appreciate, and even to practice mathematics by and for itself, but people who can and wish to do this are rare. For most of us an important value of mathematics is its applicability to other scientific disciplines. The recent fruitfulness of mathematics in this regard has already been mentioned, but this is really in the tradition of mathematics, which has repeatedly responded to other disciplines that seek to apply its theories and techniques. It should also be recognized that the sciences in their turn have stimulated the development of new fields of mathematics. Thus, we believe that students of mathematics should acquire an understanding of its wide applicability in various fields, and for this reason, applications should be emphasized in every course.

When we speak of facility with applications we mean the ability to recognize and delineate a mathematical model of a physical, social, biological, or environmental problem. Being able to solve the related mathematical problem is a skill which we also regard as important. Effective mathematics instruction must include development of the ability to attack problems by identifying their mathematical setting and then bringing appropriate mathematical knowledge to bear upon their solution.

Finally, computational skill is essential. Without it the student cannot learn to solve mathematical problems, to apply mathematics, or to appreciate even its simplest concepts and structures. Although we normally think of this skill as including speed and accuracy in applying the common algorithms of arithmetic and algebra, we should keep in mind the fact that it also includes the ability to estimate quickly an approximate result of a computation. Each course should not only provide systematic practice in computation but should also inculcate in the student the skill and habit of estimating.

This report recommends courses that we believe prospective teachers should study in order to help them achieve these objectives, both for themselves and for their future students. First we explain briefly why we believe that teachers need much more mathematical education than most of them are now getting, and why their training needs to be of a special kind in certain cases.

There have been recent improvements in the certification re-
quirements for elementary school teachers, but in our opinion they
continue to be inadequate or inappropriate in many cases. Some
states require a semester or a year of "college mathematics" without
indicating what sort of mathematics this should be. These practices
appear to be based on the assumption that little or no special train-
ing in mathematics is needed to teach in an elementary school. This
assumption has always been unrealistic, and in the present context
of rapidly changing and expanding curricula it is wholly untenable.

In some elementary schools the rudiments of algebra, informal
geometry, probability, and statistics are already being taught in
addition to arithmetic. But even if only arithmetic is taught, the
teacher needs sound mathematical training because his understanding
affects his views and attitudes; and in the classroom, the views and
attitudes of the teacher are crucial. An elementary school teacher
needs to have a grasp of mathematics that goes well beyond the con-
tent and depth of elementary school curricula.

Similarly, a Level II or III teacher's understanding of mathe-
matics must exceed, both in content and depth, the level at which he
teaches. Within the next decade it is to be expected that secondary
school teachers will be asked to teach material which many of our
present teachers have never studied.

We therefore recommend courses for all teachers which will not
only insure that they thoroughly understand the content of the courses
they must teach, but will also prepare them to discuss related topics
with able and enthusiastic students. These college courses must also
prepare teachers to make intelligent judgments about changes in con-
tent, pace, and sequence of mathematics programs for their schools,
and to have the flexibility of outlook necessary to adjust to the
curriculum changes which will surely take place in the course of
their professional careers.

THE RECOMMENDATIONS

These recommendations concern only the preparation of teachers
of elementary and secondary school mathematics. Whereas in 1961
these teachers were classified into three groups, we find it conven-
ient to use four classifications:

LEVEL I. Teachers of elementary school mathematics (grades K
through 6)

LEVEL II-E. Specialist teachers of elementary school mathematics,
coordinators of elementary school mathematics, and
teachers of middle school or junior high school mathe-
matics (roughly grades 5 through 8)

LEVEL II-J.   Teachers of junior high school mathematics (grades 7
               through 9)

LEVEL III.   Teachers of high school mathematics (grades 7 through
               12)

These classifications are to be taken rather loosely, their
interpretation depending upon local conditions of school and cur-
ricular organization. It will be noted that the various classifica-
tions overlap. This is a deliberate attempt to allow for local
variations.

The reader should note that the training for Level I teaching
is a separate program, while, except for their Level I content, the
curricula for the further levels form a cumulative sequence.

The recommendations of this report are not motivated by a
desire to meet the demands of any special program of mathematics
education or the goals of any particular planning organization. We
consider our recommendations to be appropriate for any teachers of
school mathematics, including teachers of low achievers.

Level I Recommendations

The applications of mathematics, the influence of computers,
and the changes wrought in the 1960's in the teaching of mathematics
prompt us to revise our 1961 Recommendations for the Training of
Teachers of Mathematics at all four levels. At Level I in particu-
lar we are aware that new teaching strategies designed to facilitate
and enrich learning are being adopted or are the subject of experi-
mentation. Such strategies impose on elementary school teachers
the necessity of a deeper understanding of the school mathematics
curriculum than is required by conventional teaching methods and
increase the teacher's need for knowledge of mathematics well beyond
the level at which topics are treated in the elementary classroom.

We believe that this deeper understanding can be better
achieved if mathematics is taught, and understood, from the earliest
stages as a unified subject. The function concept, for instance,
should serve as a unifying thread in elementary school mathematics,
and elementary intuitive geometry should be taught for its connec-
tions with arithmetic as well as for its own sake. The applications
of mathematics reflect its unity and offer an opportunity to illus-
trate its power. For instance, the notion of a finite sample space
in probability can be used at a very elementary level to illustrate
the idea of a set (of outcomes), and the probability of an event
can motivate the need for rational numbers. Simple statistical
problems yield practice in computing with both integers and rational
numbers as well as in applying probability theory to practical situa-
tions. Finally, the use of flowcharts helps to explain the elemen-
tary algorithms of arithmetic as well as to prepare the student for
later study of computer programming.

164

Thus, while the development of the number system should remain the core of the elementary school curriculum and of the content of Level I courses, there are other crucial topics which ought to be contained in the Level I sequence. We stress this point by listing these topics in the following recommendations.

Recommendations for Prospective Level I Teachers:

We propose that the traditional subdivision of courses for prospective elementary school teachers into arithmetic, algebra, and geometry be replaced by an integrated sequence of courses in which the essential interrelations of mathematics, as well as its interactions with other fields, are emphasized. We recommend for all such students a 12-semester-hour sequence that includes development of the following: number systems, algebra, geometry, probability, statistics, functions, mathematical systems, and the role of deductive and inductive reasoning. The recommended sequence is based on at least two years of high school mathematics that includes elementary algebra and geometry.

We further recommend that some teachers in each elementary school have Level II-E preparation. Such teachers will add needed strength to the elementary school's program.

Our suggestion of an integrated course sequence represents a very important change which these recommendations are intended to bring about, but there are certain to be questions on how this can be accomplished. We attempt in the sequel to provide our answer to such questions.

There are many ways in which to organize the appropriate material into integrated course sequences, and we encourage experimentation and diversity. Two possible sequences of four 3-semester-hour courses are described in detail in the course guides [page 175]. For reference we list their titles here:

Sequence 1 | Sequence 2

1. Number and Geometry with Applications I

1. Number Systems and Their Origins

2. Number and Geometry with Applications II

2. Geometry, Measurement, and Probability

3. Mathematical Systems with Applications I

3. Mathematical Systems

4. Mathematical Systems with Applications II

4. Functions

These sequences differ in the ordering of topics and in the degree of integration, yet both conform to our idea of an integrated course sequence. It may be helpful to discuss briefly, without any attempt

at being comprehensive or any desire to be prescriptive, the rationale which led us to these integrated sequences.

The dual role of numbers in counting and measuring is systematically exploited in each of the four-course sequences. In one direction we are led to arithmetic, in the other to geometry. The extension of the number system to include negative numbers may then be explained by reference to both the counting and measuring models. The study of rational numbers may likewise be motivated through measurement and counting, and the arithmetic of the rationals finds justification and natural applications in elementary probability theory. Geometrical considerations lead to vector addition on the line and in the plane--and then in space--and the Pythagorean Theorem leads naturally to irrational numbers.

The arithmetic of decimals can be presented as the mathematization of approximation. Also, the algorithms of elementary arithmetic lead naturally to flowcharts and to a study of the role of computers.

Length, area, and volume have computational, foundational, and group-theoretical aspects. Extensions of ideas from two to three dimensions constitute valuable experience in themselves, are useful in developing spatial intuition, and also help in understanding the nature of generalization in mathematics.

Graphs of functions of various kinds, theoretical and empirical, may be studied, incorporating intuitive notions of connectedness and smoothness.

The function concept plays an important role in an integrated curriculum. Counting, operations on numbers, measurement, geometric transformations, linear equations, and probability provide many examples of functions, and common characteristics of these examples should be noted. Ideas such as composition of functions and inverse functions may then be introduced and illustrated by algebraic and geometric examples. Of course, detailed formal discussion of the function concept should come only after examples of functions have been mentioned in various contexts in which it is useful to do so.

Similarly, the elementary notions of logic such as logical connectives, negation, and the quantifiers should be treated explicitly only after attention has been called informally to their uses in other mathematical contexts. Indirect proofs and the use of counterexamples arise naturally and may be stressed when the structure of the number systems is examined. However, in the final stages of a prospective elementary teacher's training it is useful to return to logic in a more explicit way for the purpose of summarizing the roles of inductive and deductive reasoning in mathematics and providing examples of deductive systems in geometry, algebra, number theory, or vector spaces.

Finally, the references to algorithms in the foregoing paragraphs emphasize the pervasive role of computing and algorithmic

techniques in mathematics and its applications. The use of flow-
charts for describing algorithms is becoming commonplace in the ele-
mentary school. Moreover, flowcharts are proving to be an important
educational tool in teaching elementary and secondary school students
to organize their work in problem solving. These ideas should there-
fore be encountered by a prospective teacher in his mathematics
training. We recommend that computing facilities be made available,
so that he will also have an opportunity to implement some algorithms
and flowcharts on a high-speed computer using some standard computing
language.

The course sequence should include many references to applica-
tions outside mathematics. This is self-evident for probability
theory, but it is important to stress this over the whole spectrum of
topics studied. In particular, the function concept itself provides
many opportunities to underline the significance of mathematical
formulations and methods in our study of the world around us.

At the conclusion of the course sequence the prospective
teacher should understand the rational number system and the neces-
sity, if not the method, of enlarging it to the real number system.
He should be familiar with elementary linear geometry in two and
three dimensions. In his study of the integers and the rational
numbers, he should understand the essential role played by the prop-
erties of the addition and multiplication operations and the order
relations in justifying and explaining the usual computational al-
gorithms, the factorization theory of whole numbers, and the methods
of solution of equations. He should thereby, and through experience
with algebraic structures encountered in geometry, acquire an appre-
ciation of the importance of abstraction and generalization in mathe-
matics.

He should know something of the basic concepts and the algebra
of probability theory, and he should be able to apply them to simple
problems. He should grasp the idea of an algorithmic process and
understand a bit about computers and how one programs them. We ex-
pect him to appreciate something of the role of mathematics in human
thought, in science, and in society. We hope finally that he can
learn all this in such a way that he will enjoy mathematics and the
teaching of it, and that he will desire to continue to study mathe-
matics.

Level II Recommendations

In the years since the first set of recommendations was made,
dramatic changes have taken place in the mathematics of junior high
school. These changes are in depth, as witnessed by greater empha-
sis on logic and mathematical exposition, and in breadth, as wit-
nessed by the increased amount of geometry and probability. They
make it necessary to re-examine the background needed by a teacher
at this level. Moreover, the intermediate position of the junior

high school requires of teachers at this level an appreciation of the mathematics of the elementary school as well as knowledge of the mathematics of the high school.

Finally, it seems desirable that there be two kinds of teachers in the middle school or the junior high school: those who concentrate on the transition from the elementary school and those who concentrate on the transition to the high school. For this reason we give two sets of recommendations for this level.

Level II-E Recommendations

These recommendations are for students who begin with Level I preparation and pursue further training to qualify them to be either specialist teachers of elementary school mathematics, or coordinators of elementary school mathematics, or teachers of middle school and junior high school mathematics. There should be some teachers with Level II-E preparation in each elementary school. The recommended program is:

A.   The Level I program. (A student who is already prepared for calculus may omit the course on functions of the second sequence of courses listed on page 165 .)

B.   An elementary calculus course (e.g., Mathematics 1 [page 44]). At this level all teachers need an introduction to analysis and an appreciation of the power that calculus provides.

C.   Two courses in algebra. The courses in linear and modern algebra are identical to those described under C of the Level III recommendations, page 171 .

D.   A course in probability and statistics. This course is identical to the first course under D of the Level III recommendations.

E.   Experience with applications of computing. This recommendation is identical to that under F of the Level III recommendations.

F.   One additional elective course. For example, a further course in calculus, geometry, or computing.

Level II-J Recommendations

These recommendations provide a special curriculum for the training of junior high school teachers which is slightly less extensive than that for Level III. The recommended program is:

A.   Two courses in elementary calculus (e.g., Mathematics 1 and 2, page 44). Greater emphasis on calculus is desirable for this level because teachers at the upper level of the junior high school must see where their courses lead.

B.   Two courses in algebra. The courses in linear and modern algebra are identical to those described under C of the Level III recommendations, page 171.

C.   One course in geometry. Either of the two courses described under E of the Level III recommendations will suffice.

D.   A course in probability and statistics. This course is identical to the first course described under D of the Level III recommendations.

E.   Experience with applications of computing. This recommendation is identical to that under F of the Level III recommendations.

F.   Review of the content of courses 1 and 2 of Level I (either sequence) through study or audit. There is a problem with the interface between the elementary school and the junior high school. In part this is caused by the fact that, traditionally, junior high school teachers are prepared for secondary school teaching and hence are little aware, at first, of their students' capabilities and preparation. We therefore believe that some sort of orientation to the mathematical content and spirit of the elementary school mathematics program is necessary to equip the Level II-J teacher properly. Two means have been considered to meet this need. One method would be to give an additional course in college to the prospective Level II-J teacher, a course which would be a streamlined version of courses 1 and 2 of the Level I program. On the whole, we prefer this solution although it makes the plan of study rather long. The second method would be to encourage schools to supply the new Level II-J teacher with elementary school texts to read, and to require him to visit classes and to talk with elementary school teachers, especially those in grades 4 to 6. A combination of both of these methods might prove most effective.

G.   Two elective courses.

Items A through E supply the bare essentials. Greater breadth and greater depth are both to be desired. In order to give the teacher freedom to pursue his interests, electives are suggested, with further courses in computing, analysis, algebra, and geometry having high priority. Teachers with Level III preparation can meet the requirements listed above by fulfilling the intent of F.

Level III Recommendations

Although the mathematics of the senior high school has not changed as dramatically in the past ten years as has that of the elementary school, yet there are significant directions of change which make new recommendations desirable. These are: (1) a gradual increase in the volume and depth of mathematics taught at the secondary level which brings with it an increased occurrence of calculus (with the Advanced Placement program), (2) an increasing use of computers in mathematics courses and as an adjunct in other courses, and (3) an increasing realization that applications should play a more significant role.

Our recommendations, while designed primarily to specify minimum requirements for prospective high school teachers, have also been constructed with a view to maintaining, as far as possible, comparability of standards between prospective teachers and prospective entrants to a graduate school with a major in a mathematical science. We want to maintain a freedom of choice for the student to go in either direction. While the program we recommend for prospective teachers will leave the student with a deficiency in analysis and in algebra in order to meet the CUPM recommendations for entry to graduate school, the prospective graduate student in mathematics would normally need courses in geometry and in probability and statistics to meet our recommendations for teachers. We regard it as a matter of great importance that a program for teachers should be identical to the one offered to other mathematics majors, except for a few courses peculiarly appropriate to prospective high school teachers.

Before detailing the recommendations, some remarks on the role of applications, the computer, and on the problem of teaching geometry are in order.

Every experienced teacher knows that mathematics must begin at the concrete level before it can proceed to a more theoretical or abstract formulation. It is assumed that topics in the courses under discussion will contain a judicious mixture of motivation, theory, and application. A purely abstract course for teachers would be madness, but a course in calculation with no theory would not be mathematics. In addition to including applications where possible in mathematics courses, there is a need for introducing some specific study of the lore of mathematical model building, in order to provide the framework of ideas within which specific applications can be placed in their proper perspective. The idea of a mathematical model of a "real" situation and the associated techniques and rationale of the model building process have developed as a sort of folk knowledge among mathematicians and users of mathematics, and now an effort is being directed toward making these ideas more explicit and including them in the curriculum. The course Mathematics 10 described in A General Curriculum in Mathematics for Colleges (1965) [see page 92] was such an effort, but only now are detailed descriptions of such a course appearing (see Applied Mathematics in the Undergraduate

170

Curriculum, page 705). As these efforts begin to affect the high school curriculum, where much of the material belongs, it becomes m re urgent that the future high school teacher receive appropriate preparation. Conversely, the preparation of teachers to communicate these ideas will accelerate the improved treatment of applications in high schools.

Computers have already had a phenomenal impact on the high school mathematics curriculum in supplementing, and in part replacing, traditional formal methods by algorithmic methods. As access to digital computers becomes more common, one can expect both the flavor and content of high school mathematics courses to change dramatically. In schools where such facilities are already available, it has become clear that opportunities for experimentation and creative outlet, using the computer as a laboratory device, are within the reach of many students whose mathematical ability, motivation, or background would preclude any comparable experience in a formal mathematical setting. Moreover, it has been found that certain abstract mathematical ideas are understood and appreciated more completely when experience is first obtained through the use of a computer. Algorithmic and numerical techniques should therefore be given strong consideration in all courses in which they are appropriate; and, wherever computing facilities are available, use of the computer should be a routine part of these courses.

The nature of high school geometry continues to change. Changes over the past decade have mainly been toward remedying the principal defects in Euclid's Elements that are related to the order, separation, and completeness properties of the line, but more recently there has developed an entirely new approach to geometry that links it strongly to algebra. This approach is now finding its way into the high school geometry course. A teacher should be prepared to teach geometry either in the modern Euclidean spirit or from the new algebraic point of view. Thus we are recommending that he take two geometry courses at the college level.

The minimum preparation of high school teachers of mathematics should include:

A. Three courses in calculus. Mathematics 1, 2, and 4 [page 44]
    are suitable. This recommendation assumes that the student
    has the necessary prerequisites. It is also desirable to take
    advantage of the growing role of computers in introducing
    mathematical concepts.

B. One course in real analysis. Mathematics 11 [page 93] would
    be satisfactory provided that the instructor is aware that his
    students' primary interest is teaching.

C. Two courses in algebra. One of these should treat those topics
    in linear algebra that are essential for the understanding of
    geometry and that have become crucial in applications, espe-
    cially to the social sciences. Mathematics 3 [page 55], with

careful attention to examples, would suffice. The second algebra course should be an abstract algebra course approximating Mathematics 6M [page 68]. Again, opportunities should be found to incorporate geometrical ideas that motivate and illustrate various algebraic structures (e.g., groups of symmetries, groups of transformations, rings of functions).

D.   Two courses in probability and statistics. The first of these should begin with intuitive notions of probability and statistics derived from the real world. Mathematical model building and the relationship of mathematics to the real world should be considered. Calculus may be required in the latter part of this first course. The second course will treat those additional and more advanced topics normally included in a statistics sequence. In Preparation for Graduate Work in Statistics [page 459] the Statistics Panel of CUPM has described two courses that are close to what we have in mind. Throughout the two courses, care should be taken to include an analysis of some statistical studies which have appeared on the public scene and should make explicit some of the misinterpretations that are possible. Applications (in particular, applications to decision theory) should be drawn from such fields as medicine, education, business, and politics. The range and realism of problems can be enhanced if students are able to use computers.

E.   Two courses in geometry. One course emphasizes a traditional approach by concentrating on synthetic methods and a careful study of the foundations of Euclidean geometry with a brief treatment of non-Euclidean geometry. The other course is strongly linked to linear algebra, includes an investigation of the groups of transformations associated with geometry, and is explicitly related to other parts of mathematics. Examples of such courses are given on page 86.

F.   Experience with applications of computing. This should involve learning the use of at least one higher level programming language such as BASIC or FORTRAN. For this purpose we recommend a formal course such as C1 [page 563], but the experience may also be obtained independently or in other courses that make use of computers.

G.   One course in applications. This should place heavy emphasis on mathematical models in the physical or social sciences. Examples of appropriate outlines can be found in Applied Mathematics in the Undergraduate Curriculum, page 705.

Nine of these 12 items should be included in the undergraduate program of every prospective high school teacher, namely A, B, C, F, and one course each from D and E. The remaining courses may be deferred to his post-baccalaureate study, although consideration should be given to including them among the electives in his undergraduate program. A list of possible elective courses is included at the end

of the following paragraph. Were it not for our view that an undergraduate program should permit maximum flexibility in choosing a career and as much latitude as possible for every student to express his own interests in acquiring the proper breadth in his area of concentration, we would have specified that all of the 12 items be included in the undergraduate program of a prospective teacher. Fortunately, it is now becoming commonplace for Level III teachers to continue their mathematical education at the graduate level. Indeed, this is mandatory in many instances through permanent certification requirements or through the salary schedules and policies of individual school systems.

In structuring his undergraduate mathematics program, a student will naturally choose his electives after reflection upon his career goals. If, for example, a prospective high school teacher wishes to pursue graduate study in mathematics, he will necessarily choose additional courses in algebra and analysis beyond those which we have mentioned in our recommendations for Level III teachers. We include below a partial list of electives which would suitably extend our recommendations for the training of high school mathematics teachers.

Real Variables (Mathematics 11-12, page 93)

Complex Variables (Mathematics 13, page 97)

Numerical Analysis (Mathematics 8, page 83)

Abstract Algebra (Mathematics 6M, 6L, page 65)

Geometry and Topology

Number Theory

Foundations of Mathematics

Logic and Linguistics

OTHER ASPECTS OF TEACHER TRAINING

In accordance with our charge, we have made recommendations only on the content of the teacher training curriculum. We are, however, well aware that there are other crucial aspects of a teacher's overall preparation. In discussing these matters we hasten to observe that, just as we do not believe in any sharp distinction between teacher-trainees and other students in respect to the content of their mathematics courses, so we insist that these other aspects are relevant to all mathematics instruction. We believe only that they deserve more emphasis for teacher-trainees than for other mathematics majors.

Communication is of the essence in mathematics, and prospective teachers must pay special attention to all of the ways in which mathematics is most effectively communicated. They should be led to regard mathematics as a creative activity--something which one _does_ rather than merely something which one learns. The active participation of the student in the process of discovering and communicating mathematical ideas is crucial for his real understanding. Courses should be taught in ways that foster active student involvement in the development and presentation of mathematical ideas.

Development of skills in writing and reading and speaking and listening should be an explicit part of teacher training at every stage, and not only in mathematics courses. These, like any other skills, can be developed only through constant and active involvement of the student in practices which exercise these skills. Thus, his regular courses, reading courses, clubs, or seminars should stress opportunities for two-way communication of mathematical ideas.

It is also important for teachers to continue to study and to do mathematics throughout their professional lives. This is closely related to, but goes beyond, the processes of communication mentioned above, for a willingness to grow reflects an enthusiasm that often transcends other skills in communicating mathematics.

Other aspects of communication which are not dealt with in this report are those relating to behavioral objectives and to special teaching methods and aids. While the Panel agrees that these are very important matters, it feels that they demand a much more complex effort and a totally different expertise, and might properly be the subject for another study. An excellent volume, which explores "the educational and psychological problems in the selection, organization and presentation of mathematics materials at all levels from the kindergarten through the high school," is the Sixty-ninth Yearbook of the National Society for the Study of Education, entitled "Mathematics Education."

The relationship of mathematics to other studies is another important matter not touched upon in this report except insofar as we have recommended the study of applications and mathematics. Indeed, we believe that every mathematics teacher should develop skill in other subjects which make use of mathematics.

Finally, we share a widespread concern for the special education of the culturally disadvantaged child and of the child whose achievements, for whatever reason, are below accepted standards. Such children require specially trained teachers. We do not know what form this training should take, but we feel that this is a proper concern of CUPM for the future.

COURSE GUIDES FOR LEVEL I

Introduction

Two sequences, each consisting of four courses, are outlined here in detail. One reason for presenting two different sequences is to illustrate our earlier claim that there are various ways of organizing the material. We have no desire to be prescriptive or definitive with respect either to course content or to the ordering of material. The outlines are to be construed as models only of the content and depth of coverage that we believe will be possible in the best circumstances. We do believe that the material presented approximates that which a really first-rate teacher of elementary school mathematics should know.

We regard either sequence as a way of achieving an integrated curriculum. In the first sequence the courses do not emphasize any particular single area of the traditional curriculum. Thus, arithmetic and geometry are both developed throughout the entire sequence. Each figures prominently in all four courses. In the second sequence, on the other hand, the emphasis is on number systems in the first course, geometry in the second, mathematical systems and induction in the third, and functions in the fourth. In both sequences each course contains topics from most of the areas identified as essential in the recommendation on page   . By the end of the second course in either sequence the student will have met most of the topics that we consider essential for the elementary teacher, although not at the depth or in the detail preferred. Indeed, throughout both sequences the reader must be careful to interpret the statements of topics to be covered as referring to a treatment appropriate to the level of the student, and not a definitive treatment such as would be accorded to such a topic if encountered at a higher level. Typical places where there is danger of misinterpretation are Section 5 of Course 2 in the first sequence (Operational Systems and Algebraic Structures) and Section 1 of Course 2 in the second sequence (Intuitive Non-metric Geometry).

The unification of the four courses of each sequence requires the use of a common language for the expression of mathematical ideas. For instance, the concepts of set, function, and operation are introduced early and used throughout. Logical terms are introduced and used where appropriate.

Each section of a course guide has a suggested time allocation stated as a percentage of the course. These time allocations indicate first the balance of the sections within the course, second the depth and detail of treatment of the topics listed under each section heading. Thus they should enable the reader to judge the level of treatment and avoid the danger, already referred to, of giving a more comprehensive (or, perhaps, more superficial) treatment than intended.

It must be kept in mind that a prospective teacher is profoundly influenced by what he observes and experiences as a student.

Later his own methods and philosophy of teaching will reflect that experience. Hence, it is of paramount importance that these courses be conducted in a manner which encourages active participation in mathematical discovery. Frequent and substantial assignments which expose and drive home the attendant manipulative and computational skills should also be the rule.

SEQUENCE 1

Sequence 1 consists of four courses:

1. Number and Geometry with Applications I

2. Number and Geometry with Applications II

3. Mathematical Systems with Applications I

4. Mathematical Systems with Applications II

Course 1 begins with some intuitive geometry so that the concept of a number line is available immediately. Then arithmetic and geometry are developed throughout the entire sequence; the interaction of these two areas of mathematics enriches both subjects. Nevertheless, there are occasions when each area is developed within its own context. In particular, the algebra of the rational numbers is applied extensively to the theory of probability and statistics without reference to the geometric aspects of the number line.

A feature of this sequence is the adoption, to a limited extent, of the spiral approach. Thus, certain notions, such as extensions of the number system and the group of rigid motions, reappear several times, each time at a higher level of sophistication and with enhanced mathematical knowledge at the disposal of the student. An important practical advantage of this approach is that the student who takes only two or three courses of the sequence will have met most of the important mathematical concepts. In such cases, however, the depth of understanding is less than desired.

Some simple logic is introduced where appropriate to enhance the student's understanding. At the end of the sequence the student should understand the nature of mathematical reasoning and proof.

Again, we stress the importance of interpreting the Course Guides in the spirit of the comments made in the Introduction.

Course 1. Number and Geometry with Applications I

1. Elementary Ideas of Space, Measurement, and the Number Line (20%)

2. The Rational Number System and Subsystems (60%)

3. Probability, Statistics, and Other Applications (20%)

Course 1 is concerned with the study of the rational number system. It commences with the most intuitive geometrical notions, from which attention is focused on the number line and its role as a representation of the set of whole numbers. This approach enables the arithmetical and geometrical aspects of elementary mathematics to be developed from an integrated standpoint emphasizing their complementarity. It is a particular feature of the number line that negative integers are thereby immediately suggested and readily studied; this insures that the development of the number system follows a path which is mathematically natural. The twin approach also enriches the scope for interpretation of the operations of arithmetic. At this stage these operations and their properties are motivated through physical models.

Applications of the arithmetic of the nonnegative rationals to the most intuitive ideas of probability and statistics are given. Further applications of the arithmetic should be a feature of the course.

The language should be informal, but it should be such that a transition to precise mathematical language can be naturally effected. In particular, the student should be prepared for the function concept through the use of appropriate language. Those students who have not already met the notions of sets and functions may require more explicit introductions to these concepts.

1. Elementary Ideas of Space, Measurement, and the Number Line (20%)

Intuitive development of geometric figures in the plane and space, first as idealizations of familiar objects and then as sets of points; an intuitive development of incidence relations and some simple consequences; congruence developed by use of slides and flips of models of figures leading to turns as another means of preserving congruence and with attention also to parallelism, perpendicularity, and symmetry; consideration of measurement of segments with various units and the beginning notion of approximation; informal introduction of the number line.

2.   The Rational Number System and Subsystems (60%)

Introduction of the set $W = \{0, 1, 2, 3, \ldots\}$* of whole
numbers, from sets of objects; addition in W from disjoint union,
multiplication in W from cartesian products (treated informally);
counting as the link between sets and numbers; place value systems
and decimal numeration of whole numbers (reinforced by examples with
nondecimal bases); properties of operations in W from observed
properties of operations on sets, including order properties; algo-
rithms for computation in W (include use of flowcharts); simple
closed and open mathematical sentences, including inequalities.

Coordinatization of the half-line with W; addition in W as
a vector sum, using slides of the number line; subtraction in W as
a slide to the left to introduce the set Z of integers; properties
of addition and order properties in Z; mathematical sentences in
Z. (Multiplication by negatives is delayed until the set of rationals
is developed.)

Factorization in W, prime factorization, unique prime factor-
ization, some simple divisibility criteria, the Euclidean algorithm;
multiplication and division of integers by whole numbers introduced
through experiences with the number line; greatest common divisor
and least common multiple.

Introduction of the set Q of rationals through division by
n, $n \in W$ and $n \neq 0$, as shown on the number line; change in scale
of the number line and its use in measurement; equivalence classes of
symbols for rationals; coordinatization of the line with Q; intro-
duction to the question of completeness.

Addition in Q suggested by slides of the number line; multi-
plication of a rational by a whole number suggested by slides of the
number line; multiplication by a positive rational suggested by
stretching and shrinking; multiplication by a negative integer sug-
gested by multiplication by a whole number followed by a flip; multi-
plication in Q; algorithms for computation in Q^+, properties of
addition and multiplication in Q, and order properties in Q.

---

* A glossary of symbols is included on page 202.

Decimal numeration of Q; percentages; integer exponents; scientific notation; orders of magnitude; algorithms and flowcharts for computation in Q; mathematical sentences.

3. Probability, Statistics, and Other Applications (20%)

Examples of statistical experiments in finite event spaces and their outcome sets, leading to counting procedures for determining the number of outcomes of various kinds of compound events (use tree diagrams); sampling problems with and without replacements, leading to combinatorial devices for counting samples; relative frequencies; assignment of probabilities to singleton events and to disjoint unions and intersections of events through addition and multiplication in Q^+.

Other applications, e.g., measurement, constant rate, profit and loss, expectation and risk, percentages, estimation, significant figures, and approximation.

Course 2. Number and Geometry with Applications II

1. Functions (5%)

2. The Rational Number System and Subsystems (20%)

3. Geometry (35%)

4. Real Numbers and Geometry (10%)

5. Operational Systems and Algebraic Structures (10%)

6. Probability, Statistics, and Other Applications (20%)

Course 2 is designed to give a genuine mathematical treatment of ideas introduced and studied at a more intuitive level in Course 1. The language of functions is established; this enables the solution of linear equations over Q and Z to be investigated systematically. Then the interrelationship between geometry and algebra again becomes evident in the study of symmetries, rigid motions, and sets with operations. At the same time, questions connected with measurement are studied, thus insuring that the material can be usefully applied; explicit reference to problems of approximate calculation involving large amounts of data can lead to consideration of computer programs.

The Pythagorean relation prepares the way for the introduction of irrational numbers and a preliminary discussion of real numbers.

179

The ideas here are difficult, and no attempt should be made to give complete proofs; nevertheless, the topic should be explored extensively.

Algebraic structures are defined, but studied only in familiar examples (including modular arithmetic). Further study of probability and statistics is included, beginning with a study of permutations and combinations which employs the function concept and presents systematic counting procedures.

1. Functions (5%)

The function concept (motivated by examples from Course 1), one-one and onto properties of functions; relations (motivated by examples from Course 1) with emphasis on equivalence and order relations. Binary operations as functions on cartesian products of form $X \times X$. Power sets; unions, intersections and complements as operations; the functions $2^X \to 2^Y$ and $2^Y \to 2^X$ induced by a function $X \to Y$.

2. The Rational Number System and Subsystems (20%)

Review of properties of Q with some arithmetical proofs; properties of Z; Z as an ordered integral domain; realization that Z is not closed under division, with the closure of Z leading to Q.

Coordinatization of the line with Z and then with Q; coordinatization of the plane with Z^2 and then Q^2; and coordinatization of space with Z^3 and Q^3.

3. Geometry (35%)

Review of geometric figures as idealizations of familiar objects and as sets of points in space; review of rigid motions and symmetry; review of congruence, parallelism, and perpendicularity. Groups of symmetries of an equilateral triangle and a square.

Rigid motions as functions that preserve lengths of segments; classification of rigid motions; composition and inverses; intuitive understanding of group properties of the group of rigid motions.

Review of measurement of segments with various units; principles of measurement; principles of measurement applied to length, area, angle measurement, volume; approximation; formulas for measurement related to rectangles, triangles, right prisms, pyramids.

Approximate calculation. Pythagorean relation through the formula for the area of a rectangle.

4.   Real Numbers and Geometry (10%)

Adequacy of Q for physical measurement; inadequacy of Q for representing lengths of segments demonstrated by construction of segments with irrational measures; distance between points in Q^2 (use of Pythagorean relation). Introduction of real numbers in terms of nested intervals and nonterminating decimals; location of irrational points on the number line; infinite decimals regarded as sequences of approximating rationals; informal definition of addition and multiplication in R by means of approximating terminating decimals; distance in R^2; use of unique factorization to prove the irrationality of $\sqrt{2}$, $\sqrt{3}$, etc.

Coordinatization of the line, plane, and space with the real numbers; distance in R^2. Solution of linear equations and inequalities and graphs of solution sets in R^2; solution of linear equations in R^3 as intersections of planes.

Intuitive treatment of the perimeter and area of a circle; π as an irrational number.

5.   Operational Systems and Algebraic Structures (10%)

Review of properties of the operations of addition and multiplication and the order relation in the number systems W, Z, and Q; defintions of group, ring, integral domain, and field, with examples drawn from subsets of Q, groups of rigid motions, groups of symmetries of a figure, power sets.

Arithmetic mod m as an operational system; contrasted with W, Z, and Q (closed under additive inverses, closed under multiplicative inverses when m is prime, divisors of zero when m is not prime, absence of compatible order relation); applications to the arithmetic of W (e.g., Fermat's theorem, Wilson's theorem, casting out 9's).

6.   Probability, Statistics, and Other Applications (20%)

Permutations and combinations; randomness of a sample and tests of randomness; examples of applications of random sampling

(using random number tables) to estimation of populations, quality control, etc.

Measure of central tendency in lists of data and possible measures of spread of data.

Random walks and their applications; assigning probabilities to compound events with applications; conditional probability.

Other applications, e.g., area, volume, weight, density; constructing an angle whose measure is p/q times the measure of a given angle.

Course 3. Mathematical Systems with Applications I

1. The Rational Number System (15%)

2. The Real Number System (5%)

3. Geometry (35%)

4. Functions (15%)

5. Mathematical Language and Strategy (15%)

6. Probability, Statistics, and Other Applications (15%)

In Course 3 the process of extending the number system from the whole numbers to the rationals, which has been explained and motivated in previous courses from both geometrical and arithmetical considerations, is carried out as a piece of formal algebra. Polynomials are also studied. Then algebra and geometry each provide examples of small deductive systems to illustrate the nature and power of the axiomatic method. The Pythagorean relation is available in this course, so that Euclidean geometry may be carried out in the coordinate plane. Vector notation and methods are studied, and there is a discussion of the generalization of coordinate geometry to three dimensions.

Algebraic concepts are also exemplified by the study of the group of Euclidean motions and certain subgroups, thereby enriching the notion of subgroup with concrete examples. The trigonometric, exponential, and logarithmic functions are defined and studied in their own right and in view of their applications. There are some explicit discussions in this course on mathematical methods, covering such topics as proof, conjecture, counterexample, and algorithms, together with a review of appropriate logical language. Probability theory is itself developed from an axiomatic standpoint, experience of its practical nature having been gained in previous courses.

1. **The Rational Number System** (15%)

Formal construction of Z from W and of Q from Z.

Polynomials as functions and forms; Q[x] and Z[x] as rings; the degree of a polynomial; substitution; quadratic equations.

Description of proof by induction in W with examples; application to number-theoretic properties of Z, e.g., divisibility properties, Fundamental Theorem of Arithmetic; Euclidean algorithm; similar applications to Q[x]; remainder theorem.

2. **The Real Number System** (5%)

Review of the coordinatization of the line with R; approximation of reals by rationals; addition and multiplication in R through rational approximation; the field of real numbers.

3. **Geometry** (35%)

Review of coordinate geometry of the real plane; distance; graphs of linear equations in two variables.

Plane vectors from translations; vector addition as the composition of translations; multiplication by scalars; vector equations of a line (with resulting parametric and general form of the equations); conditions for parallelism and perpendicularity in coordinate representation; appropriate generalization of these ideas to three dimensions.

Examples of groups and subgroups drawn from geometry, e.g., the group of similarities with the subgroup of rigid motions, the group of rigid motions with the subgroup of rotations about a point, the group of rotations with the subgroup of cyclic permutations of a regular polygon.

Similarities and the representation of similarities as composites of magnifications and rigid motions; similar figures. Constructing the points which separate a segment into n congruent parts.

Equations of circles and the beginning notions of trigonometric functions.

Small deductive system in plane geometry, e.g., incidence properties from postulates of incidence or some constructions from

183

postulates of congruent triangles, or some angle-measure properties from postulates of incidence, parallelism, and segment and angle measures.

4. Functions (15%)

Review of real-valued functions and their graphs; inverses of invertible functions; graph of an invertible function and its inverse.

Beginning notions of exponential functions and some of their properties; applications of these ideas, e.g., growth and decay; logarithmic functions; application of logarithmic functions to approximate calculation and construction of a slide rule.

Definitions and graphs of the trigonometric functions with emphasis on periodicity.

5. Mathematical Language and Strategy (15%)

Review of the language of connectives, the common tautologies, and the relation of some of these notions to set union, intersection, complementation, and inclusion.

Universal and existential quantifiers; denial of a mathematical statement, counterexamples.

Examples from previous sections of direct and indirect proof and of proof by induction; selected new topics to illustrate proof by induction (e.g., binomial theorem for positive integer exponents; number of zeros of a polynomial; sums of finite series); explicit contrast with inductive inference, role of hypothesis, conjecture.

Examples of algorithms and flowcharts.

6. Probability, Statistics, and Other Applications (15%)

Postulates for a discrete probability function and some consequences proved for probabilities of compound events; random walks and their applications.

Computation of measures of central tendency and variance; simple intuitive notions of statistical inference, tests of significance, frequency distributions, passage from discrete to continuous variables, normal distribution; application of statistical inference to real-life situations, e.g., opinion polls, actuarial tables, health hazards.

Course 4. Mathematical Systems with Applications II

1. Geometry (30%)

2. The Real and Complex Number Systems (10%)

3. Operational Systems and Algebraic Structures (40%)

4. Probability, Statistics, and Other Applications (20%)

Course 4 consists of a systematic study of precalculus mathematics. Linear algebra in R^2 and R^3, as vector spaces and as inner product spaces, leads on the one hand to matrix algebra and on the other to the standard trigonometric identities. The algebraic method in geometry is contrasted with the synthetic method. The real numbers R are presented as the completion of the rationals, and the extension of R to the field C of complex numbers is motivated and described.

Abstract algebra occurs in the course--a beginning study is made of abstract group theory--but emphasis is on familiar examples of the various algebraic systems, for example, the integral domain of polynomials over Z, Q, Z_p (p prime). The key notion of homomorphism of algebraic structures is introduced; among the examples treated are the logarithmic and exponential functions which are seen to be mutually inverse isomorphisms.

Frequency distributions form the main topic of the probability and statistics component; although the course remains essentially concerned with discrete probability spaces, the normal distribution is mentioned here. Applications of the preceding theory are made to problems of approximation and error.

1. Geometry (30%)

Coordinatization of space with R^3, distance in space, first-degree linear equations in three variables; vectors in space, vector addition, scalar multiples of vectors in R^2 and R^3, description of the vector spaces R^2 and R^3; norms of vectors, inner product, definition of the inner-product (or Euclidean) space R^3, relation of cosine to the inner product; definition of the vector product in R^3, triple scalar product and volume; the idea of closeness in R^3, with some of the simpler topological properties of this metric space.

Definitions of linear transformations of R^2 and R^3; orthogonal transformations; matrix representation of linear transformations of R^2 and conditions for orthogonality; matrix multiplication suggested by composition of linear transformations; representations of

rotations in the plane by orthogonal matrices, leading to the standard trigonometric identities.

Invertible linear transformations of the plane with coordinate representations; rigid motions, magnifications, and other subgroups of the group of invertible linear transformations; representation of similarities in R^2 by matrices.

Analysis of the roles of synthetic and analytic methods in geometry, e.g., properties of circles, coincidence properties of triangles.

2.   The Real and Complex Number Systems (10%)

Algebraic extensions of Q; algebraic and order properties of R; discussion of the completeness of R.

Extension of the real number system to the field C of complex numbers; failure of order relations in C; graphical representation of C in R^2.

3.   Operational Systems and Algebraic Structures (40%)

System of polynomial forms over Q, its integral domain properties, factorization; Euclidean algorithm and appropriate flow diagram; factor theorem; elementary theory of polynomial equations; comparison with theory for polynomials over Z_p (p prime), Z.

Exponents, extension of exponential functions over Q to functions over R, with graphs; rational functions over Q, over Z_p (p prime); Newton's method of approximating zeros of polynomials (no differential calculus) and appropriate flow diagram.

Subgroups; Lagrange's theorem; applications to elementary number theory (Fermat's theorem and Euler's theorem); commutative groups, quotient groups of commutative groups; application to Z_n.

Homomorphisms of algebraic structures with many examples; identification of those which are one-one, onto; defintion of isomorphism as invertible homomorphism; one-one and onto homomorphisms are isomorphisms; isomorphic systems, e.g., Z_4 and rotational symmetries of a square, the positive reals under multiplication and the real numbers under addition.

4.   <u>Probability, Statistics, and Other Applications</u> (20%)

Review of sample spaces, probability functions, random walks; discrete binomial distributions; statistical inference and tests of significance; other frequency distributions with applications, e.g., rectangular, Poisson; normal distribution (treated descriptively).

Application of the number system Q to problems in scaling, ratio, proportion, variation; approximation, errors in approximation, errors in sums, errors in products; Bayesian inference.

SEQUENCE 2

Sequence 2 consists of four courses:

1.   Number Systems and Their Origins

2.   Geometry, Measurement, and Probability

3.   Mathematical Systems

4.   Functions

Each course contains topics from most of the areas identified in the general description of the Level I Recommendations (algebra, the function concept, geometry, mathematical systems, number systems, probability, deductive and inductive reasoning). While interrelationships among these topics are explored in the spirit of an integrated curriculum, each of the four courses, nevertheless, has a special emphasis or focus. The emphasis in the first course is on number systems, the second on geometry, the third on mathematical systems, and the fourth on functions.

Although the special character of each course can be suggested in a few words, it is a mistake to assume that any course is narrowly defined by its title. In fact, by the end of the second course the student will have met the <u>full breadth</u> of topics considered essential for the elementary teacher. He will not, however, have reached the <u>depth</u> of understanding desired.

Again, we stress the importance of interpreting the Course Guides in the spirit of the comments made in the Introduction.

Course 1. Number Systems and Their Origins

1. Sets and Functions (15%)

2. Whole Numbers (45%)

3. First Look at Positive Rational Numbers (10%)

4. First Look at Integers (5%)

5. The Systems of Integers and Rationals (25%)

Course 1 features integration of arithmetic and algebra with
supplementary assistance from geometry. The number line is thought
of as a convenient device for representing numbers, order, and opera-
tions. Rational numbers are introduced in the context of comparing
discrete rather than continuous sets (though brief reference is also
made to the rational line). Algebraic similarities and differences
between the number systems are emphasized. In particular, the sys-
tems of integers and rationals are studied in parallel. Algorithms,
flowcharts, and manipulative rules for the various number systems are
not only justified by referring to physical or schematic models but
also are seen as consequences of the algebraic structural properties
of the number systems. The whole number system receives much atten-
tion, as its algebraic properties (and its algorithms) recur in only
slightly altered form in the systems of integers, rationals, and
reals.

1.   Sets and Functions (15%)

Review, at an _intuitive_ level, of the basic concepts associated

with sets and functions in order to establish the language and nota-

tion that will be used throughout the course. (For most students

this will be a review of things they have seen repeatedly since

junior high school.) Set concepts covered are: membership, inclu-

sion, and equality for sets; various ways of describing sets (rosters,

set-builder notation, Venn diagrams); special subsets that often lead

to misunderstanding (empty set, singletons, the universal set); com-

mon operations on sets (intersection, union, complementation, carte-

sian product); illustration of the above concepts in various ways

from real objects and from geometry.

The connection between set operations and logical connectives;

for example, "or," "and," and "not" are related to union and inter-

section and complement while the inclusion relation "$A \subset B$" is

related to the implication "$x \in A \Rightarrow x \in B$." (In this first intro-

duction of logic, the treatment should be very brief and informal,

but the language is necessary for subsequent use.)

The major function concepts to be covered include an intuitive rule-of-assignment definition; various ways of specifying this rule (arrow diagram, table, graph, set of ordered pairs, formula); notions of domain and range; input-machine-output analogy; one-one and onto properties; one-one correspondences between finite sets and between infinite sets; brief look at composition and inverses with a view toward later ties to rational arithmetic. (Real and geometric examples should be used).

2. Whole Numbers (45%)

Whole numbers are motivated by a desire to specify the "size" of finite sets, numerals and numeration systems by the inadequacy of verbal "symbols" for numbers; the Hindu-Arabic numeration system contrasted with historical and modern artificial numeration systems in order to emphasize the roles of base and place value; order among whole numbers related to the process of counting and to the existence of one-one or onto functions between finite sets; order in W represented schematically by the position of points on a whole number line. In the construction of this "line" the concept of congruent point pairs arises naturally.

The operations of addition and multiplication related, with counting as the link, to the set operations of union and cartesian product (e.g., the use of multiplication in determining the area of a rectangle); multiplication also related to repeated addition and to determining the number of outcomes in a multi-stage experiment (the product formulas for C_r^n and P_r^n might be illustrated); addition and multiplication represented schematically in the usual vector fashion (slides and stretches) on the number line.

The algorithms of whole-number arithmetic justified initially (as in the elementary classroom) by reference to manipulating and grouping finite sets. (A flow diagram for division via repeated subtraction can be given.) The whole numbers with their operations and order now viewed as a mathematical system, the algebraic properties of which are motivated by reference to finite sets and set operations; the algorithms of whole number arithmetic re-examined from an internal

point of view and justified on the basis of notational conventions and fundamental algebraic structural principles. The importance of estimating products and quotients should be emphasized as the algorithms are studied.

3.   First Look at Positive Rational Numbers (10%)

Fractions motivated by a desire to compare two finite sets. If a probabilistic flavor is desired, compare a set of favorable outcomes with a set of possible outcomes; if a more conventional approach is desired, "ratio" situations can be used. With fractions viewed as operators, addition continues to correspond, in a sense, to disjoint union, while multiplication corresponds to composition; fractions represented schematically as points or vectors on a number line, and the operations viewed vectorially; rules for manipulating fractions motivated initially from physical or schematic representations. Rational numbers appear as abstractions of equivalence classes of fractions, and some algebraic properties of the system of rational numbers can be motivated by physical examples; the algebraic structure of Q^+ is not explored in detail. (The embedding of W in Q^+ considered briefly with a light touch.) A careful structural investigation deferred until the full system Q appears.

4.   First Look at Integers (5%)

Integers suggested by some real situation, e.g., profit-loss, up-down, etc.; addition corresponds to an operation in the given situation; multiplication by a positive integer considered as repeated addition. Again the number line is used as a schematic representation; the various uses of the symbol " - " clarified; some algebraic properties identified and motivated; the embedding of the whole numbers in Z introduced, but treated only lightly.

5.   The Systems of Integers and Rationals (20%)

Following a brief review of the concepts of open sentence, variable, replacement set, truth set, equation, and solution set (or perhaps only the last two), a systematic, parallel exposition of the algebraic structures of Q and Z in terms of solutions to equations is possible (-a represents the unique solution to $a + x = 0$,

190

1/a represents the unique solution to ax = 1 (a ≠ 0),

b - a = b + (-a), b/a = b X 1/a); all the familiar rules for
manipulating minus signs and fractions follow. The two important
unique representations of rationals--as fractions in lowest terms
and as (all but one type of) repeating decimals--can be illustrated
and computational rules for (finite) decimals justified; several non-
repeating infinite decimals described. The work on fractions in
lowest terms will involve a certain amount of number theory, which
should be done on an _ad_ _hoc_ basis. Review of the concepts of divis-
ibility and prime; the Fundamental Theorem of Arithmetic illustrated
and then assumed. (In Course 3 this principle may be proved.)

Course 2. Geometry, Measurement, and Probability

1.   Intuitive Nonmetric Geometry (25%)

2.   Intuitive Metric Geometry (25%)

3.   Probability (20%)

4.   Further Geometry (20%)

5.   The Real Number System (10%)

In Course 2 the system of positive rational numbers reappears
in two new contexts: in the context of geometric measurement, where
continuous sets are being compared, and in the context of probability,
where discrete sets are being "measured." Thus the system of posi-
tive rational numbers and the concept of measurement act as unifying
threads. But the major blocks of new content covered are in geometry
and probability.

Many opportunities for tying together these areas present
themselves. For example, in the initial work in geometry which in-
evitably is concerned with establishing terminology and notation,
combinatorial problems can be inserted to make the content more
interesting. Later a probabilistic technique for approximating π
could be given. Also, while studying probability, geometric repre-
sentations can be given for many situations. For example, random
processes are simulated by spinners, and experiments involving re-
peated trials are represented by random walks.

The geometry and the probability in this course are presented
in a rather intuitive, nondeductive fashion. The main purpose here
is to present the elementary facts in these areas, not to investi-
gate their logical structure. Course 3 re-examines both areas from
a more rigorous, deductive point of view.

The field of real numbers is also discussed to some extent in this second course.

1.   Intuitive Nonmetric Geometry (25%)

Geometry viewed as the study of subsets of an abstract set, called space, whose elements are called points; the subsets are called geometric figures; drawing conventional pictures of points, lines, and planes suggests incidence relations. Some of the logical connections between various incidence properties explored. (The 4-point geometry might be introduced here, but axiomatics should not receive much emphasis in this course.) The standard terminology associated with incidence (collinear, coplanar, concurrent, parallel, skew, ...) reviewed in a combinatorial context (e.g., into how many pieces is the plane partitioned by n lines no three of which are concurrent and no two of which are parallel?). Further geometric figures (half lines; rays; open, closed, and half-open segments; half planes, half spaces, plane and dihedral angles and their interiors and exteriors) defined in terms of the basic figures--points, lines, planes; the set operations; the intuitively presented notions of (arcwise) connectivity and betweenness. The standard notation for these figures reviewed, using combinatorial problems for motivation (e.g., how many angles are "determined" by n points no three of which are collinear?).

The ideas of polygonal and nonpolygonal curves, simple curves, simple closed curves, and the interior of a simple closed curve presented intuitively along with a few other topological and geometrical concepts such as dimension of a figure, boundary of a figure, convexity; polygons, polyhedra, and Euler's formula.

Congruence introduced intuitively in this nonmetric setting as meaning same size and shape. At this first contact, the notions of measurement and distance avoided. The natural development of ideas seems to be: congruence which leads to a process of measuring which in turn suggests the existence of a distance function. Perhaps here, but probably more appropriately in Course 3, a definition of congruence in terms of distance can be given. Congruence of segments, angles, and other plane and spatial figures, with perpendicularity introduced in terms of congruence of adjacent angles.

192

Congruence in the plane viewed in terms of intuitive notions
of rigid motions of the plane (slides, turns, flips, and their com-
positions). Symmetries of figures in terms of invariant point sets
and rigid motions. The composition of rigid motions is a rigid
motion and the inverse of a rigid motion is a rigid motion. The
group concept introduced to tie together algebra and geometry. A
fuller treatment of transformation geometry is suggested in Course 4.

2. Intuitive Metric Geometry (25%)

The process of measuring described in terms of filling up the
set to be measured with congruent copies of a unit and counting the
number of units used. Illustrated for segments, angles, and certain
plane and spatial figures. Integrally nonmeasurable figures (with
respect to a given unit) introduced and the positive rationals used
as operators endowed with stretching-shrinking or replicating-
partitioning powers. The rational number line reinterpreted in
terms of segments and lengths, briefly showing the existence of
rationally nonmeasurable segments and the real number line; a non-
repeating infinite decimal exhibited and the theorem on decimal
representation of irrationals recalled. The assignments of numbers
to figures viewed as functions; observation that such measure func-
tions are additive and invariant under congruence. The domain of
segment measure functions extended to the domain of polygonal curves
by additivity; perimeters computed. The additivity property applied
to partitioning techniques for finding area; some familiar area and
volume formulas derived (triangles, parallelograms, prisms, pyramids).
The formula $A = \ell \times w$ for rectangles with irrational dimensions
illustrated by drawing inscribed and circumscribed rectangles with
rational dimensions. Plausible limiting arguments presented for
circles and spheres; irrationality of π. The angle-sum theorem for
triangles verified experimentally and then extended to convex n-
gons by triangulation; the subsequent results about the various
angle measures in regular polygons applied to making ruler-protractor
drawings. Use of these measuring instruments suggests investigation
of practical versus ideal measurement. Various units of length,
area, angle, volume measurements and conversion factors relating

193

them; the inevitability of approximation in practical measurement and the usage of such terms as "greatest possible error," "precision," "accuracy," "relative error." The various notational conventions in use for reporting how good an approximation is: significant digits, ± notation, interval of measure, scientific notation.

3.   Probability (20%)

Various single and multi-stage experiments with discrete sample spaces considered and represented geometrically (trees, walks, spinners); large sample spaces and events "counted" using permutation and combination techniques. The terminology--sample space, outcome, event--compared with the terminology of geometry--space, point, figure; the assignment of probabilities to events compared with the assignment of lengths, areas, etc., to geometric figures. Both involve a comparison of two sets; the rational numbers are the indicated algebraic system. A priori assignment of probabilities (from shape of die, partitioning of spinner, constituency of urn, ...) compared with a posteriori assignment (long-range stability of relative frequency of events). The assignment of probabilities to events in terms of the point probabilities of their constituent outcomes, in the finite case, leading to additivity of probability measures; Venn diagrams used to illustrate the connection between set operations and logical connectives and to suggest the useful formulas

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad \text{and} \quad P(A) = 1 - P(A').$$

Conditional probability and independent events; problems involving multiplication along the branches of a tree. The connection between C_r^n and the number of paths of a certain kind in the plane; the formula $C_r^n p^r (1 - p)^{n-r}$ for r successes in n repeated trials deduced and used; Pascal's Triangle (if not described earlier). The work on probability should include many exercises, as this may be the first contact with the subject for many prospective elementary school teachers.

4.   Further Geometry (20%)

The simplest straightedge-compass constructions reviewed and related to the parallel postulate and congruence conditions for

194

triangles (some proving of triangles congruent appropriate here).
Each straightedge-compass construction technique compared with a
ruler-protractor drawing technique for the same figure. Some plausi-
bility argument for the Pythagorean Theorem, perhaps the one sug-
gested by this sketch.
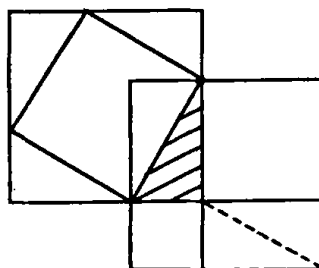
The simplified congruence condition for right triangles stated.
Some work on square roots is appropriate here: a proof, based on the
Fundamental Theorem of Arithmetic, that $\sqrt{n}$ is irrational or a whole
number; an algorithm or two for computing rational approximations to
$\sqrt{n}$; a remark that the existence of square roots in R but not in Q
is a tipoff that R must have some extra fundamental property that
Q does not have; geometric construction of a segment with irrational
length, with respect to a given unit. Projection techniques for
drawing similar triangles reviewed. Applications to scale drawing
and to straightedge-compass construction of the rational number line.
Similarity conditions for triangles analogous to the congruence con-
ditions for triangles; for the case of right triangles the "AAA"
similarity condition reduces to just "A." This suggests the calcula-
tion of trigonometric ratios and their use in indirect measurement.
The Law of Cosines as a generalization of the Pythagorean Theorem.

5.   The Real Number System (10%)

     A brief review of the properties of the reals in some nonformal

195

way--e.g., R includes Q; enjoys the same basic properties of +, X, < that Q does but has one more property, namely, the least upper bound property. Use of this property to suggest but not prove the existence of $\sqrt[n]{a}$ for all $a \in R^+$ and all $n \in Z^+$. Some work with rational exponents.

Course 3. Mathematical Systems

1. The System of Whole Numbers (25%)

2. Fields (25%)

3. Geometry (25%)

4. Probability-Statistics (25%)

In this course certain portions of algebra, geometry, and probability are studied more deeply in a systematic, deductive way. Proof receives more emphasis than in Courses 1 and 2. Some of the concepts of logic itself receive explicit treatment, along with new results in algebra, geometry, and probability.

1. The System of Whole Numbers (25%)

The algebraic and order properties of W reviewed; the well-ordering principle introduced. Symbol $a|b$ defined; proofs of some simple divisibility theorems such as $a|b \Rightarrow a|bc$, $a|b$ and $a|c \Rightarrow a|(b+c)$. Various simple divisibility criteria of the base ten numeration system derived. Primes and prime factorization; the sieve of Eratosthenes; checking for prime divisors of n only up to $\sqrt{n}$; Euclid's theorem and Wilson's theorem. Unsolved problems such as Goldbach's conjecture and the twin primes problem. Other interesting odds and ends, e.g., figurate, deficient, abundant, and perfect numbers.

The important concepts of common and greatest common divisor (GCD) introduced and the existence, uniqueness, and linear combination expressibility theorems for GCD derived. Techniques for finding the GCD (by listing all divisors, by the Euclidean Algorithm, from known prime factorizations); the least common multiple (LCM), its existence and uniqueness proved, its relation to the GCD, and several techniques for finding it. (A flow diagram for the Euclidean

196

Algorithm is appropriate here.) The concept of relative primeness and the lemma stating that $p|ab \Rightarrow p|a$ or $p|b$ (p prime), leading to a proof of the Fundamental Theorem of Arithmetic. Whether a rigorous proof of this theorem should be given using the well-ordering principle or mathematical induction, is debatable. The use of the Fundamental Theorem in reducing fractions and in demonstrating the existence of irrationals. Euler's φ-function might be defined and the theorems of Euler and Fermat illustrated.

2. <u>Fields</u> (25%)

Field defined using Q and R as prototypes. Subtraction and division in a field defined and their usual properties derived; Z_p (p prime) defined and shown to be a field; various properties of subtraction and division illustrated again in this context. Possibly enough group theory interposed (Lagrange's theorem, order of element theorem) to prove the theorems of Euler and Fermat. In the context of solving a linear equation over a field, several concepts of logic can be studied; statement, equality, variable, open sentence, reference set, truth set; the unsolvability of $x^2 = 2$ over Q contrasts with its solvability over R; the completeness property of R recalled. The statement of this property depends on the concept of order; the definition of an ordered field abstracted from familiar properties of Q and R. Simple order properties deduced; Z_p shown to be unorderable (in any decent sense). The logical connectives and their relation to the set operations, within the context of solving inequalities over (say) R. Equivalence of open sentences; equivalence transformations. The Archimedean and density properties for R derived; a short excursion into limits (optional). The intermediate value theorem cited, behavior of polynomials for large $|x|$ illustrated; existence and uniqueness of positive n^{th} roots deduced.

3. <u>Geometry</u> (25%)

Incidence axioms suggested by the <u>Euclidean</u> plane and space stated as abstract axioms and exhibited in a finite model; simple incidence theorems proved and interpreted in both models. Illustrate

197

in the context of segments how the natural genesis of concepts: congruence (superposition) → process of measurement → distance function, can be reversed in a formalization of geometry: postulated distance function → congruence defined in terms of it. The intuitive notion of betweenness used to define rays and segments; betweenness also defined in terms of distance; the ruler postulate; proofs of a few elementary betweenness properties. (The depth to which this Birkhoff-SMSG approach is carried is a matter of taste. For the future elementary teacher it might be more appropriate to do most of the deductive work in the spirit of Euclid, pointing out from time to time an implicit betweenness or existence assumption.) Possible subjects for short deductive chains include: triangle congruence and straightedge-compass constructions; parallels, transversals, and angle sums; area postulates and an area proof of the Pythagorean Theorem.

Some flavor of other modern approaches to geometry, with attention restricted to the plane: coordinatization of the plane, vector addition and scalar multiplication of points, lines as subspaces and their cosets, vector and standard equations for lines; Pythagorean Theorem and its converse, Law of Cosines, perpendicularity, dot product, norm, distance; isometry, orthogonal transformation, matrix representation of linear transformations, classification of orthogonal transformations, decomposition of an isometry into a translation and an orthogonal transformation. Alternatively, a coordinate-free study of transformation groups.

4.   Probability-Statistics (25%)

Review of the "natural" development of the terminology and basic concepts of outcome, sample space, event, point probability function, and probability measure; axioms for a probability measure. The possible backward rigorization in probability of the intuitive notion of "equally likely outcomes" as "outcomes having the same probability" compared with the backward rigorization in geometry of "congruent segments" as "segments having the same length." It might be worthwhile to digress in more generality on equivalence relations, partitions, functions, and preimages. After a few deductions from the

axioms $[P(\emptyset) = 0, \quad P(A') = 1 - P(A), \quad P(A \cup B) = P(A) + P(B)$
$- P(A \cap B)]$, less emphasis is placed on deduction and further proba-
bilistic concepts and techniques are presented. The idea of <u>simula-</u>
<u>tion</u> by urn models, balls in cells, spinners, and the use of random
number tables illustrated through a wide variety of problems. Permu-
tations and combinations reviewed. More formal attention to combina-
torial identities such as

$$2^n = (1 + 1)^n = \sum_{i=0}^{n} C_i^n, \qquad C_{k-1}^{n-1} + C_k^{n-1} = C_k^n,$$

$$\sum_{i=j}^{n-1} C_j^i = C_{j+1}^n.$$

More work on conditional probability, repeated trials, and random
walks. Simple problems in hypothesis testing (e.g.: Ten tosses of a
coin result in eight heads. With what confidence can you reject the
hypothesis that the coin is honest?). Elementary expectation prob-
lems where expectation is thought of simply as a weighted average
(e.g.: How much should one expect to win on a roll of a die if the
payoff when n turns up is n^2 dollars?). The same problem posed
using a nonsymmetric spinner instead of a die. Common distribution
functions; measures of spread.

<u>Course 4. Functions</u>

1. Real Functions (10%)

2. Algebraic Functions (20%)

3. Exponential and Logarithmic Functions (15%)

4. Transformations and Matrices (20%)

5. Trigonometric Functions (15%)

6. R^2 and the Dot Product (20%)

The purpose of this course is two-fold: to present a satisfy-
ing culmination to the four-course sequence and to prepare the stu-
dent to continue in mathematics with a calculus course such as Mathe-
matics 1 [page 44]. Both of these goals are met in the context of a

course centered around the function concept. Preparation for calculus is accomplished by studying special real functions, namely, the elementary functions; study of special functions of the plane (affine transformations) provides an appropriate dénouement of the four-course sequence by bring together arithmetic, algebra, and geometry in a transformation approach to plane geometry.

1. Real Functions (10%)

Brief review of the general concept of function from both the rule-of-assignment and ordered-pair points of view; specialization to the case where the domain and range are real numbers. Simple examples of functions--some artificial, some from science or business. Graphing and reading graphs. Graphs of real functions make it easier to think of them as mathematical objects in their own right, subject to operations as are other mathematical objects. Addition, subtraction, multiplication, division, and composition of functions viewed graphically as well as algebraically. Various additive and multiplicative groups of functions. One could look for rings, vector spaces, or even algebras of functions if that much abstract algebra is available.

2. Algebraic Functions (20%)

Real functions specialized to polynomial functions with emphasis on linear and quadratic functions. Slope and equations of straight lines, zeros of polynomials, and the factor theorem. Graphical interpretation of linear functions in the plane; geometric interpretation of linear functions on the number line in terms of stretches, shrinks, slides, and flips. (This suggests considering analogous functions of the plane and presents a natural opening for the discussion of general mappings of the plane and then the special types of mappings which are important for geometry, namely, translations, rotations, magnifications, and reflections. A discussion of transformation geometry may be included at this point.)

Increasing and decreasing real functions; there is no analogous concept for plane functions since the plane is not ordered. The completeness of the real number system and the Intermediate Value Theorem done at an intuitive level; invertible functions, both in the context of functions of the plane and in the context of real functions.

For real functions this leads to work with roots and rational exponents. Quadratic equations and various explicit algebraic functions.

3.   Exponential and Logarithmic Functions (15%)

It is not reasonable to give a rigorous development of exponential functions. After adequate study of a^x for x rational and after some geometric motivation, the existence of a^x for x real should be assumed. The "laws of exponents" need emphasis. Other isomorphisms and homomorphisms recalled. Graphs of exponential functions and combinations thereof; logarithm functions defined as inverses of exponential functions, their properties derived from the properties of exponential functions; a minimal amount of computational work with common logarithms.

4.   Transformations and Matrices (20%)

Having just completed computational work with some real functions, one can naturally ask whether various functions of the plane can also be given a concrete numerical representation. This leads to linear algebra and the study (in R^2) of vectors, dependence, independence, basis, linear transformation, matrix representation of linear transformations, matrix multiplication, and transformation composition.

5.   Trigonometric Functions (15%)

The sine and cosine functions introduced by recalling the trigonometric ratios (Course 2); their definitions in terms of the winding function. The other trigonometric functions defined, graphs drawn, and questions of periodicity and invertibility entertained. Rotation matrix derivation of the addition formulas for sines and cosines. Proof of other trigonometric identities. The Pythagorean Theorem and its converse recalled and the Law of Cosines proved as a generalization.

6. R^2 and the Dot Product (20%)

The dot product motivated by the Law of Cosines as a measure of perpendicularity. The chain of ideas from dot product through length to metric traced. The central role played by this metric in

currently popular axiomatic developments of geometry. The geometric
and algebraic significance of the determinant function for 2 X 2
matrices.

Glossary of Symbols

SYMBOL	MEANING
W	$\{0, 1, 2, \ldots\}$, the set of whole numbers
Z	The set of integers
Q	The set of rational numbers
R	The set of real numbers
C	The set of complex numbers
Z^+, Q^+, R^+	The set of positive elements of Z, Q, R, respectively
Z_n	The set of integers modulo n
$x \in A$	The element x belongs to the set A
$A \subset B$	The set A is a subset of set B
A X B	$\{(x,y): x \in A \text{ and } y \in B\}$
A^2	A X A
A^3	A X A X A
$\Rightarrow$	implies
$a \mid b$	a divides b
GCD	Greatest common divisor
LCM	Least common multiple
C_r^n	$\dfrac{n!}{r!(n - r)!}$
P_r^n	$\dfrac{n!}{(n - r)!}$
2^A	The set of all subsets of A
A[x]	The set of all polynomials in one indeterminate with coefficients in A