TEACHER EDUCATION IN ALGEBRA*

C. C. MACDUFFEE, University of Wisconsin

For many years we at the University of Wisconsin have been experimenting with our curriculum in algebra. Both the preparation of the students and the development of the subject itself have changed so rapidly that it has not been possible to develop a curriculum which would remain satisfactory for more than a few years. It is therefore with a feeling of little confidence that I approach this subject. I can only relate some of our experiences and experiments under the supposition that other colleges are having similar experiences, and state a few of our conclusions for whatever they may be worth.

The very spotty preparation in algebra of most of our freshmen is a major item among our troubles. Even among those students who enter the engineering college, the second year of algebra in high school is rapidly disappearing. Even more serious is the lack of competence in arithmetic and algebra of those who claim a year's preparation. Until fashions in Education swing back, there seems to be nothing effective that we can do to stem this trend. To entering engineering students we give an entrance test on elementary skills, and those students who are completely incompetent are required to spend a preliminary semester in a no-credit course where they study high school algebra. In the College of Letters and Science we give an entrance test but have no non-credit course.

Let us admit, then, that the first semester of college mathematics is largely remedial and that very little is actually learned beyond the basic skills. This course is primarily aimed at providing the tools for analytic geometry and the calculus and a minute examination of its contents does not seem to be important.

It is in the courses in analytic geometry and the calculus that the student becomes adept in algebraic manipulations. The grossly incompetent students are no longer present, and imperceptibly the problems have involved more and more complex algebraic operations so that at long last the student has acquired some skill in algebraic manipulation.

The question which we have come here to discuss is the proper content of a course or courses in algebra for a student who has just completed elementary calculus. There are many possibilities, of which I list three, because they seem to represent most common procedures:

1. A problem-solving course in permutations and combinations, probability *etc.* with advanced skills as the prime objective.

2. A development of some of the more elegant topics in classical algebra, a course in which skills and theory are kept in balance.

3. An introduction to abstract algebra.

* A lecture delivered at the Symposium on Teacher Education in Mathematics, August 27. 1952, in Madison, Wisconsin.

Before deciding which of these procedures to follow, one must examine carefully the objectives of the course. Some students at this level are aiming to take graduate work in mathematics, but more of them are not. In fact, we have graduate and undergraduate students from almost all scientific departments in the university and we cannot afford to ignore their demands for algebraic skills. It is this diversity of objectives which makes the problem difficult and calls for a compromise answer.

Since physics has become such an important science, both undergraduate and graduate students in physics have reached a level where they can be treated with mathematics majors. Most of them now enter into the spirit of a mathematics course and enjoy the theory as well as the applications. They present no particular problem.

It is frequently otherwise with advanced undergraduate and graduate students from other departments. I have some superior students from chemistry and others who are superior only in their attitude toward the Mathematics Department. They object to learning anything whose immediate application to chemistry is not evident. It is difficult to say how far our obligation to such students goes.

A few engineering students elect this course, but by and large these students keep pretty closely to their differential equations and mechanics. There is a scattering of students from agriculture, biology and the other sciences but these students are not numerous and they are inclined to be not very good. Their principal interest in algebra is as a prelude to statistics.

Approximately half of our students in junior algebra are mathematics majors. They fall sharply into two groups, graduate students who enter with a deficiency in algebra, and undergraduates working for a teacher's certificate. The latter sometimes find the competition a little rough.

To this list we might add a few students who are interested in the actuarial examinations, and a group which bids fair to increase in the next few years, namely students who are interested in computing machines.

If the University were large enough so that each of these groups could be segregated and an algebra course tailored to the individual needs of each group, the problem of the curriculum would be a small one. However, this course usually runs in at most two sections so that this cannot be done.

Probably very few mathematicians would be willing to go as far as some of our applied brothers seem to desire and omit all theory. In fact, if such an experiment were tried, I am sure that the students would become so confused that it would have to be abandoned. On the other hand, a straight course in mathematical theory is out of place with a group such as we have in our classes. A skillful teacher will know how to keep these features in balance.

With such a diversified clientele it seems undesirable to deviate too greatly from the traditional course, which I assume to be the theory of equations. There are certain definite advantages in this course. It enables one to teach skills in computing that are of definite value to students in the sciences and at the same time develop elegant processes of reasoning that were not employed in the calculus.

While I do not believe that the student should be plunged into a course in abstract algebra at this stage in his development, I do believe that the shadow of abstract algebra falls upon this course and indicates certain changes in its organization and development which should be made.

The course which I have been giving at Wisconsin for the last couple of years is still entitled the Theory of Equations, but might more properly be called the Theory of Polynomials. This approach seems to unify the somewhat scattered topics in the theory of equations, and to give a deeper insight into the subject which is particularly valuable to those who go on in algebra and to those who contemplate teaching algebra.

You may not agree with me in bringing in about a week of the theory of numbers, but I have found it desirable, and after all the only way to check a pedagogical theory is to try it out. Only a little of the theory of numbers is required, the definitions of primes and units, scales of notation, the greatest common divisor algorithm, and the unique factorization theorem. The ideas are here presented in their simplest form free of computational difficulties. Later the corresponding concepts and theorems must be proved for polynomials. Just as a matter of experience I have found that students have trouble with this topic if it is first presented to them by way of polynomials. The introduction of this bit of number theory seems necessary in order to teach the polynomial theory regardless of its intrinsic interest.

After this bit of number theory it is easy to attack the problem of finding the integral solutions of an equation having integral coefficients, and the rational solutions of an equation having rational coefficients. Let us consider for a moment the theorem that if an equation with integral coefficients has a rational solution, when this solution is expressed in lowest terms the numerator is a divisor of the constant term of the equation. The proof depends upon the theorem in number theory that if a number divides a product and is relatively prime to one of the factors, it must divide the other factor. This students are ordinarily asked to accept as obvious, but I have seen some able students quite disturbed by it. It is proved in our bit of number theory and students having had this week of number theory really seem to understand the proof.

The plan which I have been following is to start with polynomials over the rational field, later take up polynomials over the real field, and finally polynomials over the complex field. It is well to call attention to the properties of a field, but it is not necessary to call them postulates or to introduce the notion of an abstract field. Other fields, if introduced at all, are postponed until late in the course.

The high points in the theory of polynomials are the Euclid algorithm, the unique factorization theorem, the representation of one polynomial as a polynomial in powers of a second, and the properties of the derivative in regard to multiple zeros. All of these results hold for every coefficient field but can first be introduced for polynomials over the rational field. At this point the decomposition of a rational function into a sum of partial fractions can be rigorously established. The students are familiar with the process from integral calculus and some of them are astonished when you point out that the universality of the method had not been proved to them.

Most books on the theory of equations scrupulously avoid assuming any knowledge of the calculus on the part of the student. There may be some historical reason for this assumption, but it is no longer valid in American colleges. It is very unusual for any student to elect the theory of equations before he has had his first semester of calculus, and when it *is* done, it should not be. I am accustomed to use the derivative freely and I thus avoid the awkward situation of having to explain why the limit process is out of bounds in an algebra course.

I have always maintained that an intuitive approach should precede an abstract approach, and I feel that this is particularly true in the introduction of the real numbers. All the mystery can be taken out of the real numbers if one can show the student that the existence of an approximation process with arbitrarily small error establishes the existence of the real number. This is no place to go into the intricacies of real variable theory, but an intuitive grasp of the meaning of the real numbers is not beyond the legitimate objectives of a course in the theory of equations.

I would leave a consideration of the complex field until the last instead of introducing it at the beginning, as has lately become fashionable. To deflate the complex field from its paramount position in mathematics is one of the objectives of abstract algebra.

One of the oddities of modern convention is the introduction by many books of the complex numbers by means of Hamilton's number pairs. This is certainly in the spirit of modern abstract algebra, but it is such an isolated bit of abstraction that it seems decidedly out of place in a book that is otherwise purely intuitive. The reason why the complex numbers are frequently so introduced, and not the rational numbers nor the real numbers, is simply a matter of history. Hamilton introduced the complex numbers in this manner in 1835, but the work of Steinitz in 1910 is still probably too new for similar incorporation. In fact, I prefer to do neither, but to keep the presentation on an intuitive level at this point. The complex numbers can be nicely introduced by means of their correspondence with the points of the plane.

I would like to put in a plea here for a few days devoted to symmetric functions of the roots of an equation. Graduate students nearly always tell me that it was omitted from their course in the theory of equations on the grounds that it is of no practical value. This is a point which I am unwilling to concede. Not only is it essential for more advanced work in algebra, but it shows with great clarity how the coefficients of an equation determine the roots without showing favoritism to any one root or group of roots.

The rest of the course is pretty conventional, featuring the factorization of polynomials with real and complex coefficients. This leads to the solutions of the cubic and quartic equations in terms of radicals. Since determinants are not a part of this course, some time is devoted to the solution of linear systems of equations by applying elementary transformations to them. This treatment of systems of equations is fast gaining in popularity. It is never more laborious than to use determinants, and in the irregular cases it is quicker and more satisfactory. A systematic treatment of systems of higher degree is also included.

It is possible to make this course more intuitive and less formal than is sometimes done. One can teach methods rather than formulas, as in the instance of finding bounds for the roots of an equation, and methods stay with a student better than a formula.

In the seventeen years that I have been at Wisconsin, this junior algebra course has vacillated between a one semester course and a two semester course several times. Now I think it has permanently become a two semester course, the second semester being devoted to matrices and determinants. Until recently it has been difficult to find a text well suited to such a course, but the publishers now seem to be erupting with many such books. This seems to indicate that other institutions than Wisconsin are planning courses along these same lines.

The recent growth of the theory of matrices in public esteem has been remarkable, but not at all surprising. Matrices are now employed in statistics, differential equations, and in computational mathematics of various kinds, to mention only a few applications. Students from other departments are coming to us with requests for work in matrices and we are having about as many students in this second semester of algebra as in the first. Probably well over half of our mathematics majors in Education take it after they have completed the Theory of Equations course.

A course in matrices at the junior level must be given with some restraint. This is not the point for an instructor or text book writer to unload all he knows. The concept of matrix must be motivated at the beginning until the student has got some idea of what it is all about.

As I have usually given the course, the first few weeks are given over to the treatment of determinants of matrices. This treatment differs very little from the old-fashioned development of determinants except in the statement of the theorems. A determinant is a number and does not have rows or columns. The matrix has the rows and columns, but a matrix is not equal to an ordinary number. Thus the accurate statement of a well-known theorem is that if two rows of a matrix are proportional, the determinant of the matrix is zero.

It seems quite natural to most students to set up the detached coefficients of a linear system of equations, and to call this array a matrix. Elementary operations on the equations, which replace the given system of equations by an equivalent system, correspond to elementary row operations on the matrix which replace it by a row-equivalent matrix. By a simple sequence of row operations every linear system may be replaced by one whose matrix is in Hermite form, whereupon the system of equations is solved. The process is applicable regardless of the number of equations or of unknowns, of whether the system is consistent or inconsistent, of whether there is one solution or infinitely many.

Without doubt the next most important topic to be treated is quadratic form theory. This has familiar geometric motivations, and important applications to statistics, relativity and other fields. There are some computational difficulties, so that the first problems to be worked should be chosen so as to minimize these difficulties. The problem should be formulated in matric notation as well as in the notation of quadratic forms, and pretty soon the student will be dealing with matrices as abstract entities.

This is a most excellent place to point out the significance of the concept of field, and the fundamental differences between the rational, real and complex fields. The concept of invariant is well illustrated by the rank and signature of a quadratic form.

Custom now calls for a treatment of orthogonal matrices, and the orthogonal reduction of a real symmetric matrix to diagonal form. For third order matrices this is the principal-axis transformation of solid analytic geometry. To students who have clawed rather ineffectually at this problem in a geometry class, the complete solution of the problem is revealing.

Such then, is the content of our course 115 which most candidates for the teacher's certificate with major in mathematics take at Wisconsin. It is a fairly substantial course, but I believe I speak for my colleagues when I say that we are never entirely satisfied with it. There are so many things that it does not contain that it would be good for a teacher of mathematics to know. It is true that a few of the concepts of abstract algebra are worked into it, such as field, ring and group, but so much is omitted that a modern student of mathematics should know.

You are probably about to ask the question, why do we not put the prospective teachers into a separate section and give them abstract algebra. We have tried it without much success. Most of our Education students are not in the genius class and they absorb the abstract point of view very slowly. Also they do not have the basic algebraic facility nor the fund of illustrative examples to make the subject meaningful. It looks as if we shall have to be content to keep our junior algebra course at a somewhat old-fashioned level and surreptitiously to work in as many abstract concepts as the material allows.

But to my mind the situation indicates pretty strongly that a well qualified teacher of mathematics in high school should have a master's degree or at least a little summer work beyond the bachelor's degree. This is not a revolutionary suggestion, but the really radical and communistic suggestion that I am going to make is that the master's work should include at least one course in the subject which the candidate teaches. Every summer we have thousands of teachers in our Summer School, a dozen or so of whom ever go inside North Hall.

We have for about eight years offered courses in the Summer Session expressly for high school teachers of mathematics. They are entitled Foundations of Arithmetic and Foundations of Algebra, and are given in alternate years. We had believed that as time went on these courses would attract an increasing clientele, but the classes remain small while hundreds of teachers go next door for courses in the Principles of Education.

The course in Foundations of Arithmetic exposes the student to some modern mathematics. The natural numbers are introduced by means of the Peano postulates, the rational integers as pairs of natural numbers, the rational numbers as pairs of rational integers, the real numbers as sequences of rational numbers, the complex numbers as pairs of real numbers, and finally the real quaternions as tetrads of real numbers. This is Hamilton's idea carried to a logical conclusion. Within this framework there is opportunity to take up scales of notation other than the decimal scale, proof of the rule of casting out nines, and a critical examination of all of the elementary operations of arithmetic including cube root. The student is treated to proofs of all the fundamental laws of arithmetic including the fact that zero times one is zero. It is true that the proof is not one that the teacher can show to an elementary student, but at least the teacher knows that it is true.

The course has been very successful with those whom we have been able to persuade to take it. The point of view has been rather lofty but we have not required a very high level of attainment on the part of the student teacher. We have felt that it was sufficient to get across the general idea without requiring the memorization of too many details. In fact, there was frequently marked enthusiasm by the class when we finally proved some point which had previously bothered these teachers when they had attempted to present the subject to their own elementary classes.

The other course, given in alternate summers with the Foundations of Arithmetic, was aimed at strengthening the foundations of elementary algebra. In my own opinion it was not as good a course as the first. Almost without exception the students registering for the course had not had the theory of equations, so that we were up against the situation described earlier · · · no technical competence. As a result this course has overlapped considerably with the theory of equations.

I am afraid that in this course I have spent some time presenting antidotes to some of the ideas which the students have picked up in methods courses—not at the University of Wisconsin, I hasten to add. I shall take just one example. In a methods course, there are various distinct methods for solving a quadratic equation: factoring, completing the square, and using the formula. But from the point of view of a person who is interested in fundamental principles, there is just one: a product is zero if and only if one of its factors is zero.

I wonder if it is really a good idea to break down a subject such as algebra into a large number of steps, methods and processes. Perhaps it is the only way with very unintelligent students. But with students of medium intelligence or above, I think it is a step in the wrong direction. The number of fundamental principles in algebra is small and if these principles can be kept to the forefront, and the student made to realize that he can rely on common sense, the whole subject should appear vastly simplified. I have often said that if I were teaching a class in high school mathematics I would want a room with large blank wall spaces on either side. In one of these spaces I would have printed in large letters, "A product is equal to zero if and only if at least one of the factors is zero." In the other blank space I would print, "A fraction is equal to zero if and only if its numerator is zero and its denominator is not zero." A large amount of algebraic knowledge is locked up in these two statements.

One of the commonest misconceptions that I have found among these teachers relates to the concept of axiom or postulate. I suppose this is quite natural, for the concept is tricky and many books are quite wrong or, let us say, out of date in their usage. Actually there are very few genuine postulates in elementary non-geometric mathematics. The Postulates of Peano or some equivalent assumptions seem to be necessary for the introduction of the positive integers, but after that it is simply a matter of defining a new type of number in terms of those already defined, defining operations upon them, and then of finding out what properties these new numbers and operations possess. To call the commutative law or the distributive law an axiom is dishonest. Either it is true or not true for the numbers with which you work, and you have no choice whether you will assume it or not. The fact that you are not able to present the proof to an elementary class does not justify you in calling it an axiom. It is better to merely say that the numbers with which you are working have this property. If some bright student challenges your statement, tell him honestly that the proof is too tricky for presentation to an elementary class.

May I put in a little political plug before closing? In these days when the master's degree is coming more and more to be required for advancement in the profession, is it too much to ask that half of the work toward the degree be in the subject which the candidate expects to teach or in some related subject? How infinitely more capable and more inspiring would a teacher of algebra be if he were not only conversant with classical algebra as exemplified by a year's work in algebra beyond the calculus, but if he had also taken a graduate course in abstract algebra. I do not wish to imply, of course, that a teacher should start presenting abstract algebra to his pupils in high school, but I am sure that he would radiate confidence and enthusiasm to a degree that would make him a trusted and admired teacher.

Frankly I do not know how such a program could be put over. I assume that those high school teachers of mathematics who take work in mathematics in summer sessions are among the best and most enthusiastic in the high school field. But most of them are not very good scholars. They are not as good as our run-of-mine juniors. Probably they once were, but after being out of the atmosphere for a few years they have deteriorated. What the others are like I hate to think.

But I do not see how we are going to persuade high school teachers of mathematics to take more mathematics in the summer time. School boards frequently encourage teachers to attend summer school, but the subjects that they shall take is seldom specified or, if it is specified, it is in the Principles of Education.

And a more delightful summer vacation may be had by by-passing subject matter courses.

Possibly some good might be done if the books on How to Teach Mathematics were of a little higher calibre. An Educational expert who does not know any mathematics beyond elementary algebra is not in a position to write such a book. It is perhaps also true that a research mathematician is not able to restrain himself sufficiently to keep the book within the comprehension of his audience, in which case his book will simply not be read. What we really need is another J. W. Young and an American David Hilbert who will write elegant but elementary expositions for the benefit of those who are not very experienced in the field, men who are not afraid to tell less than they know about a subject in fear that some book reviewer will misjudge the depth of their scholarship.

ON FACTORIZATION OF POLYNOMIALS

J. B. KELLY, Michigan State College

1. Introduction. The problem of determining, in a finite number of steps, the factorization over the rational field of a given numerical polynomial has been solved by Kronecker. His method is based on the fact that, when the polynomials $f(x)$ and $g(x)$ have integral coefficients, $f(x)$ can be divisible by $g(x)$ only if $f(n)$ is divisible by $g(n)$ for any integer n. Even when one takes advantage of the improvements in Kronecker's method which have been developed by various writers, its application usually involves lengthy and cumbersome calculations. In this paper we present an alternative procedure which focuses attention on the coefficients of the polynomial, rather than on its values at the integers. We believe that our method requires less effort than Kronecker's.

2. The ∇-functions.

Let

$$f(x) = x^n + \sum_{j=1}^{n} (-1)^j s_j x^{n-j}$$

be a polynomial with rational integral coefficients and leading coefficient one. s_j is the elementary symmetric function of order j of the roots $r_1, r_2, \cdots, r_n$ of $f(x)$. We define the integer $\nabla_{j,k} f(x)$ by means of the equation,

$$\nabla_{j,k} f(x) = \prod \sigma_j(r_1', r_2', \cdots, r_k'),$$