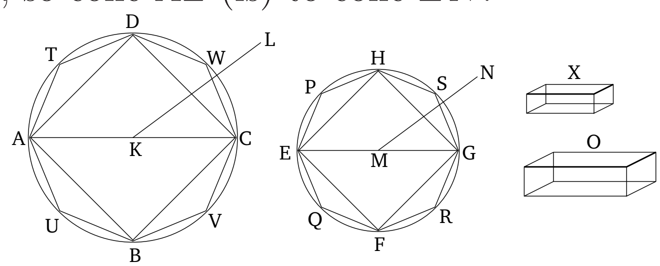
Book 12
Proposition 11

Cones and cylinders having the same height are to one another as their bases.

Let there be cones and cylinders of the same height whose bases [are] the circles $ABCD$ and $EFGH$, axes KL and MN, and diameters of the bases AC and EG (respectively). I say that as circle $ABCD$ is to circle $EFGH$, so cone AL (is) to cone EN.



For if not, then as circle $ABCD$ (is) to circle $EFGH$, so cone AL will be to some solid either less than, or greater than, cone EN. Let it, first of all, be (in this ratio) to (some) lesser (solid), O. And let solid X be equal to that (magnitude) by which solid O is less than cone EN. Thus, cone EN is equal to (the sum of) solids O and X. Let the square $EFGH$ have been inscribed in circle $EFGH$ [Prop. 4.6]. Thus, the square is greater than half of the circle [Prop. 12.2]. Let a pyramid of the same height as the cone have been set up on square $EFGH$. Thus, the pyramid set up is greater than half of the cone, inasmuch as, if we circumscribe a square about the circle [Prop. 4.7], and set up on it a pyramid of the same height as the cone, then the inscribed pyramid is half of the circumscribed pyramid. For they are to one another as their bases [Prop. 12.6]. And the cone (is) less than the circumscribed pyramid. Let the circumferences EF, FG, GH, and HE have been cut in half at points P, Q, R, and S. And let HP, PE, EQ, QF, FR, RG, GS, and SH have been joined. Thus, each of the triangles HPE, EQF, FRG, and GSH is greater than half of the segment of the circle about it [Prop. 12.2]. Let pyramids of the same height as the cone have been set up on each of the triangles HPE, EQF, FRG, and GSH. And, thus, each of the pyramids set up is greater than half of the segment of the cone about it [Prop. 12.10]. So, (if) the remaining circumferences are cut in half, and straight-lines are joined, and pyramids of equal height to the cone are set up on each of the triangles, and this is done continually, then we will (eventually) leave some segments of the cone (the sum of) which is less than solid X [Prop. 10.1]. Let them have been left, and let them be the (segments) on HPE, EQF, FRG, and GSH. Thus, the remaining pyramid whose base is polygon $HPEQFRGS$, and height the same as the cone, is greater than solid O [Prop. 6.18]. And let the polygon $DTAUBVCW$, similar, and similarly laid out, to polygon $HPEQFRGS$, have been inscribed in circle $ABCD$. And on it let a pyramid of the same height as cone AL have been set up. Therefore, since as the (square) on AC is to the (square) on EG, so polygon $DTAUBVCW$ (is) to polygon $HPEQFRGS$ [Prop. 12.1], and as the (square) on AC (is) to the (square) on EG, so circle $ABCD$ (is) to circle $EFGH$ [Prop. 12.2], thus as circle $ABCD$ (is) to circle $EFGH$, so polygon $DTAUBVCW$ also (is) to polygon $HPEQFRGS$. And as circle $ABCD$ (is) to circle $EFGH$, so cone AL (is) to solid O. And as polygon $DTAUBVCW$ (is) to polygon $HPEQFRGS$, so the pyramid whose base is polygon $DTAUBVCW$, and apex the point L, (is) to the pyramid whose base is polygon $HPEQFRGS$, and apex the point N [Prop. 12.6]. And, thus, as cone AL (is) to solid O, so the pyramid whose base is $DTAUBVCW$, and apex the point L, (is) to the pyramid whose base is polygon $HPEQFRGS$, and apex the point N [Prop. 5.11]. Thus, alternately, as cone AL is to the pyramid within it, so solid O (is) to the pyramid within cone EN [Prop. 5.16]. But, cone AL (is) greater than the pyramid within it. Thus, solid O (is) also greater than the pyramid within cone EN [Prop. 5.14]. But, (it is) also less. The very thing (is) absurd. Thus, circle $ABCD$ is not to circle $EFGH$, as cone AL (is) to some solid less than cone EN. So, similarly, we can show that neither is circle $EFGH$ to circle $ABCD$, as cone EN (is) to some solid less than cone AL.

So, I say that neither is circle $ABCD$ to circle $EFGH$, as cone AL (is) to some solid greater than cone EN.

For, if possible, let it be (in this ratio) to (some) greater (solid), O. Thus, inversely, as circle $EFGH$ is to circle $ABCD$, so solid O (is) to cone AL [Prop. 5.7 corr.]. But, as solid O (is) to cone AL, so cone EN (is) to some solid less than cone AL [Prop. 12.2 lem.]. And, thus, as circle $EFGH$ (is) to circle $ABCD$, so cone EN (is) to some solid less than cone AL. The very thing was shown (to be) impossible. Thus, circle $ABCD$ is not to circle $EFGH$, as cone AL (is) to some solid greater than cone EN. And, it was shown that neither (is it in this ratio) to (some) lesser (solid). Thus, as circle $ABCD$ is to circle $EFGH$, so cone AL (is) to cone EN.

But, as the cone (is) to the cone, (so) the cylinder (is) to the cylinder. For each (is) three times each [Prop. 12.10]. Thus, circle $ABCD$ (is) also to circle $EFGH$, as (the ratio of the cylinders) on them (having) the same height.

Thus, cones and cylinders having the same height are to one another as their bases. (Which is) the very thing it was required to show.