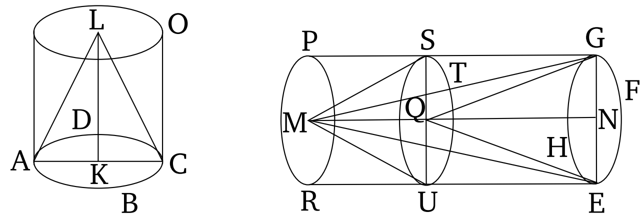
Book 12
Proposition 15

The bases of equal cones and cylinders are reciprocally proportional to their heights. And, those cones and cylinders whose bases (are) reciprocally proportional to their heights are equal.



Let there be equal cones and cylinders whose bases are the circles $ABCD$ and $EFGH$, and the diameters of (the bases) AC and EG, and (whose) axes (are) KL and MN, which are also the heights of the cones and cylinders (respectively). And let the cylinders AO and EP have been completed. I say that the bases of cylinders AO and EP are reciprocally proportional to their heights, and (so) as base $ABCD$ is to base $EFGH$, so height MN (is) to height KL.

For height LK is either equal to height MN, or not. Let it, first of all, be equal. And cylinder AO is also equal to cylinder EP. And cones and cylinders having the same height are to one another as their bases [Prop. 12.11]. Thus, base $ABCD$ (is) also equal to base $EFGH$. And, hence, reciprocally, as base $ABCD$ (is) to base $EFGH$, so height MN (is) to height KL. And so, let height LK not be equal to MN, but let MN be greater. And let QN, equal to KL, have been cut off from height MN. And let the cylinder EP have been cut, through point Q, by the plane TUS (which is) parallel to the planes of the circles $EFGH$ and RP. And let cylinder ES have been conceived, with base the circle $EFGH$, and height NQ. And since cylinder AO is equal to cylinder EP, thus, as cylinder AO (is) to cylinder ES, so cylinder EP (is) to cylinder ES [Prop. 5.7]. But, as cylinder AO (is) to cylinder ES, so base $ABCD$ (is) to base $EFGH$. For cylinders AO and ES (have) the same height [Prop. 12.11]. And as cylinder EP (is) to (cylinder) ES, so height MN (is) to height QN. For cylinder EP has been cut by a plane which is parallel to its opposite planes [Prop. 12.13]. And, thus, as base $ABCD$ is to base $EFGH$, so height MN (is) to height QN [Prop. 5.11]. And height QN (is) equal to height KL. Thus, as base $ABCD$ is to base $EFGH$, so height MN (is) to height KL. Thus, the bases of cylinders AO and EP are reciprocally proportional to their heights.

And, so, let the bases of cylinders AO and EP be reciprocally proportional to their heights, and (thus) let base $ABCD$ be to base $EFGH$, as height MN (is) to height KL. I say that cylinder AO is equal to cylinder EP.

For, with the same construction, since as base $ABCD$ is to base $EFGH$, so height MN (is) to height KL, and height KL (is) equal to height QN, thus, as base $ABCD$ (is) to base $EFGH$, so height MN will be to height QN. But, as base $ABCD$ (is) to base $EFGH$, so cylinder AO (is) to cylinder ES. For they are the same height [Prop. 12.11]. And as height MN (is) to [height] QN, so cylinder EP (is) to cylinder ES [Prop. 12.13].

Thus, as cylinder AO is to cylinder ES, so cylinder EP (is) to (cylinder) ES [Prop. 5.11]. Thus, cylinder AO (is) equal to cylinder EP [Prop. 5.9]. In the same manner, (the proposition can) also (be demonstrated) for the cones. (Which is) the very thing it was required to show.