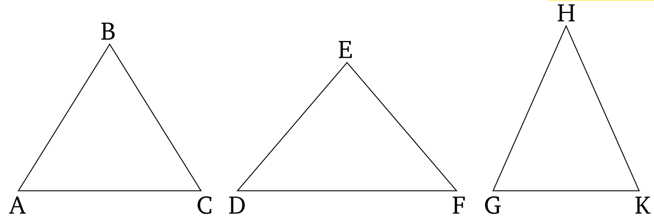
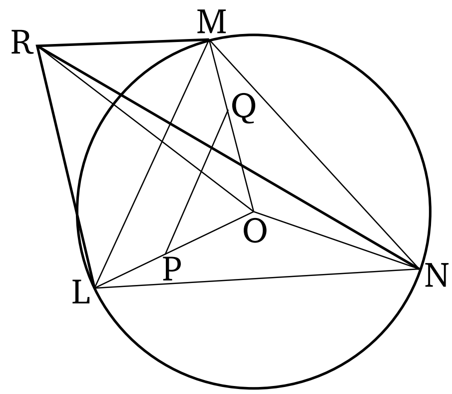
Book 11
Proposition 23

To construct a solid angle from three (given) plane angles, (the sum of) two of which is greater than the remaining (one, the angles) being taken up in any (possible way). So, it is necessary for the (sum of the) three (angles) to be less than four right-angles [Prop. 11.21].



Let ABC, DEF, and GHK be the three given plane angles, of which let (the sum of) two be greater than the remaining (one, the angles) being taken up in any (possible way), and, further, (let) the (sum of the) three (be) less than four right-angles. So, it is necessary to construct a solid angle from (plane angles) equal to ABC, DEF, and GHK.

Let AB, BC, DE, EF, GH, and HK be cut off (so as to be) equal (to one another). And let AC, DF, and GK have been joined. It is, thus, possible to construct a triangle from (straight-lines) equal to AC, DF, and GK [Prop. 11.22]. Let (such a triangle), LMN, have be constructed, such that AC is equal to LM, DF to MN, and, further, GK to NL. And let the circle LMN have been circumscribed about triangle LMN [Prop. 4.5]. And let its center have been found, and let it be (at) O. And let LO, MO, and NO have been joined.



I say that AB is greater than LO. For, if not, AB is either equal to, or less than, LO. Let it, first of all, be equal. And since AB is equal to LO, but AB is equal to BC, and OL to OM, so the two (straight-lines) AB and BC are equal to the two (straight-lines) LO and OM, respectively. And the base AC was assumed (to be) equal to the base LM. Thus, angle ABC is equal to angle LOM [Prop. 1.8]. So, for the same (reasons), DEF is also equal to MON, and, further, GHK to NOL. Thus, the three angles ABC, DEF, and GHK are equal to the three angles LOM, MON, and NOL, respectively. But, the (sum of the) three angles LOM, MON, and NOL is equal to four right-angles. Thus, the (sum of the) three angles ABC, DEF, and GHK is also equal to four right-angles. And it was also assumed (to be) less than four right-angles. The very thing (is) absurd. Thus, AB is not equal to LO. So, I say that AB is not less than LO either. For, if possible, let it be (less). And let OP be made equal to AB, and OQ equal to BC, and let PQ have been joined. And since AB is equal to BC, OP is also equal to OQ. Hence, the remainder LP is also equal to (the remainder) QM. LM is thus parallel to PQ [Prop. 6.2], and (triangle) LMO (is) equiangular with (triangle) PQO [Prop. 1.29]. Thus, as OL is to LM, so OP (is) to PQ [Prop. 6.4]. Alternately, as LO (is) to OP, so LM (is) to PQ [Prop. 5.16]. And LO (is) greater than OP. Thus, LM (is) also greater than PQ [Prop. 5.14]. But LM was made equal to AC. Thus, AC is also greater than PQ. Therefore, since the two (straight-lines) AB and BC are equal to the two (straight-lines) PO and OQ (respectively), and the base AC is greater than the base PQ, the angle ABC is thus greater than the angle POQ [Prop. 1.25]. So, similarly, we can show that DEF is also greater than MON, and GHK than NOL. Thus, the (sum of the) three angles ABC, DEF, and GHK is greater than the (sum of the) three angles LOM, MON, and NOL. But, (the sum of) ABC, DEF, and GHK was assumed (to be) less than four right-angles. Thus, (the sum of) LOM, MON, and NOL is much less than four right-angles. But, (it is) also equal (to four right-angles). The very thing is absurd. Thus, AB is not less than LO. And it was shown (to be) not equal either. Thus, AB (is) greater than LO.

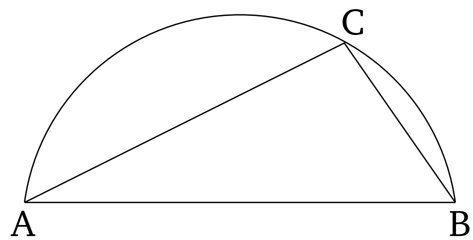
So let OR have been set up at point O at right-angles to the plane of circle LMN [Prop. 11.12]. And let the (square) on OR be equal to that (area) by which the square on AB is greater than the (square) on LO [Prop. 11.23 lem.][1]. And let RL, RM, and RN have been joined.

And since RO is at right-angles to the plane of circle

[1]Since this is a reference to the lemma included with this Proposition, no arrow is given on the graph.

LMN, RO is thus also at right-angles to each of LO, MO, and NO. And since LO is equal to OM, and OR is common and at right-angles, the base RL is thus equal to the base RM [Prop. 1.4]. So, for the same (reasons), RN is also equal to each of RL and RM. Thus, the three (straight-lines) RL, RM, and RN are equal to one another. And since the (square) on OR was assumed to be equal to that (area) by which the (square) on AB is greater than the (square) on LO, the (square) on AB is thus equal to the (sum of the squares) on LO and OR. And the (square) on LR is equal to the (sum of the squares) on LO and OR. For LOR (is) a right-angle [Prop. 1.47]. Thus, the (square) on AB is equal to the (square) on RL. Thus, AB (is) equal to RL. But, each of BC, DE, EF, GH, and HK is equal to AB, and each of RM and RN equal to RL. Thus, each of AB, BC, DE, EF, GH, and HK is equal to each of RL, RM, and RN. And since the two (straight-lines) LR and RM are equal to the two (straight-lines) AB and BC (respectively), and the base LM was assumed (to be) equal to the base AC, the angle LRM is thus equal to the angle ABC [Prop. 1.8]. So, for the same (reasons), MRN is also equal to DEF, and LRN to GHK.

Thus, the solid angle R, contained by the angles LRM, MRN, and LRN, has been constructed out of the three plane angles LRM, MRN, and LRN, which are equal to the three given (plane angles) ABC, DEF, and GHK (respectively). (Which is) the very thing it was required to do.



Lemma

And we can demonstrate, thusly, in which manner to take the (square) on OR equal to that (area) by which the (square) on AB is greater than the (square) on LO. Let the straight-lines AB and LO be set out, and let AB be greater, and let the semicircle ABC have been drawn around it. And let AC, equal to the straight-line LO, which is not greater than the diameter AB, have been inserted into the semicircle ABC [Prop. 4.1]. And let CB have been joined. Therefore, since the angle ACB is in the semicircle ACB, ACB is thus a right-angle [Prop. 3.31]. Thus, the (square) on AB is equal to the (sum of the) squares on AC and CB [Prop. 1.47]. Hence, the (square) on AB is greater than the (square) on AC by the (square) on CB. And AC (is) equal to LO. Thus, the (square) on AB is greater than the (square) on LO by the (square) on CB. Therefore, if we take OR equal to BC then the (square) on AB will be greater than the (square) on LO by the (square) on OR. (Which is) the very thing it was prescribed to do.