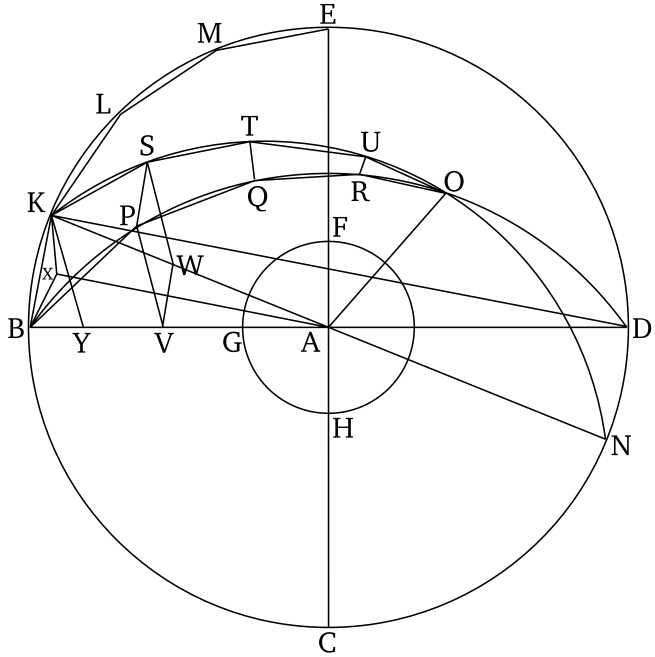
Book 12
Proposition 17

There being two spheres about the same center, to inscribe a polyhedral solid in the greater sphere, not touching the lesser sphere on its surface.



Let two spheres have been conceived about the same center, A. So, it is necessary to inscribe a polyhedral solid in the greater sphere, not touching the lesser sphere on its surface.

Let the spheres have been cut by some plane through the center. So, the sections will be circles, inasmuch as a sphere is generated by the diameter remaining behind, and a semi-circle being carried around [Def. 11.14]. And, hence, whatever position we conceive (of for) the semi-circle, the plane produced through it will make a circle on the surface of the sphere. And (it is) clear that (it is) also a great (circle), inasmuch as the diameter of the sphere, which is also manifestly the diameter of the semi-circle and the circle, is greater than all of the (other) [straight-lines] drawn across in the circle or the sphere [Prop. 3.15]. Therefore, let $BCDE$ be the circle in the greater sphere, and FGH the circle in the lesser sphere. And let two diameters of them have been drawn at right-angles to one another, (namely), BD and CE. And there being two circles about the same center—(namely), $BCDE$ and FGH—let an equilateral and even-sided polygon have been inscribed in the greater circle, $BCDE$, not touching the lesser circle, FGH [Prop. 12.16], of which let the sides in the quadrant BE be BK, KL, LM, and ME. And, KA being joined, let it have been drawn across to N. And let AO have been set up at point A, at right-angles to the plane of circle $BCDE$. And let it meet the surface of the (greater) sphere at O. And let planes have been produced through AO and each of BD and KN. So, according to the aforementioned (discussion), they will make great circles on the surface of the (greater) sphere. Let them make (great circles), of which let BOD and KON be semi-circles on the diameters BD and KN (respectively). And since OA is at right-angles to the plane of circle $BCDE$, all of the planes through OA are thus also at right-angles to the plane of circle $BCDE$ [Prop. 11.18]. And, hence, the semi-circles BOD and KON are also at right-angles to the plane of circle $BCDE$. And since semi-circles BED, BOD, and KON are equal—for (they are) on the equal diameters BD and KN [Def. 3.1]—the quadrants BE,

BO, and KO are also equal to one another. Thus, as many sides of the polygon as are in quadrant BE, so many are also in quadrants BO and KO equal to the straight-lines BK, KL, LM, and ME. Let them have been inscribed, and let them be BP, PQ, QR, RO, KS, ST, TU, and UO. And let SP, TQ, and UR have been joined. And let perpendiculars have been drawn from P and S to the plane of circle $BCDE$ [Prop. 11.11]. So, they will fall on the common sections of the planes BD and KN (with $BCDE$), inasmuch as the planes of BOD and KON are also at right-angles to the plane of circle $BCDE$ [Def. 11.4]. Let them have fallen, and let them be PV and SW. And let WV have been joined. And since BP and KS are equal (circumferences) having been cut off in the equal semi-circles BOD and KON [Def. 3.28], and PV and SW are perpendiculars having been drawn (from them), PV is [thus] equal to SW, and BV to KW [Props. 3.27, 1.26]. And the whole of BA is also equal to the whole of KA. And, thus, as BV is to VA, so KW (is) to WA. WV is thus parallel to KB [Prop. 6.2]. And since PV and SW are each at right-angles to the plane of circle $BCDE$, PV is thus parallel to SW [Prop. 11.6]. And it was also shown (to be) equal to it. And, thus, WV and SP are equal and parallel [Prop. 1.33]. And since WV is parallel to SP, but WV is parallel to KB, SP is thus also parallel to KB [Prop. 11.1]. And BP and KS join them. Thus, the quadrilateral $KBPS$ is in one plane, inasmuch as if there are two parallel straight-lines, and a random point is taken on each of them, then the straight-line joining the points is in the same plane as the parallel (straight-lines) [Prop. 11.7]. So, for the same (reasons), each of the quadrilaterals $SPQT$ and $TQRU$ is also in one plane. And triangle URO is also in one plane [Prop. 11.2]. So, if we conceive straight-lines joining points P, S, Q, T, R, and U to A then some solid polyhedral figure will have been constructed between the circumferences BO and KO, being composed of pyramids whose bases (are) the quadrilaterals $KBPS$, $SPQT$, $TQRU$, and the triangle URO, and apex the point A. And if we also make the same construction on each of the sides KL, LM, and ME, just as on BK, and, further, (repeat the construction) in the remaining three quadrants, then some polyhedral figure which has been inscribed in the sphere will have been constructed, being contained by pyramids whose bases (are) the aforementioned quadrilaterals, and triangle URO, and the (quadrilaterals and triangles) similarly arranged to them, and apex the point A.

So, I say that the aforementioned polyhedron will not touch the lesser sphere on the surface on which the circle FGH is (situated).

Let the perpendicular (straight-line) AX have been drawn from point A to the plane $KBPS$, and let it meet the plane at point X [Prop. 11.11]. And let XB and XK have been joined. And since AX is at right-angles to the plane of quadrilateral $KBPS$, it is thus also at right-angles to all of the straight-lines joined to it which are also in the plane of the quadrilateral [Def. 11.3]. Thus, AX is at right-angles to each of BX and XK. And since AB is equal to AK, the (square) on AB is also equal to the (square) on AK. And the (sum of the squares) on AX and XB is equal to the (square) on AB. For the angle at X (is) a right-angle [Prop. 1.47]. And the (sum of the squares) on AX and XK is equal to the (square) on AK [Prop. 1.47]. Thus, the (sum of the squares) on AX and XB is equal to the (sum of the squares) on AX and XK. Let the (square) on AX have been subtracted from both. Thus, the remaining (square) on BX is equal to the remaining (square) on XK. Thus, BX (is) equal to XK. So, similarly, we can show that the straight-lines joined from X to P and S are equal to each of BX and XK. Thus, a circle drawn (in the plane of the quadrilateral) with center X, and radius one of XB or XK, will also pass through P and S, and the quadrilateral $KBPS$ will be inside the circle.

And since KB is greater than WV, and WV (is) equal to SP, KB (is) thus greater than SP. And KB (is) equal to each of KS and BP. Thus, KS and BP are each greater than SP. And since quadrilateral $KBPS$ is in a circle, and KB, BP, and KS are equal (to one another), and PS (is) less (than them), and BX is the radius of the circle, the (square) on KB is thus greater than double the (square) on BX.† Let the perpendicular KY have been drawn from K to BV.‡ And since BD is less than double DY, and as BD is to DY, so the (rectangle contained) by DB and BY (is) to the (rectangle contained) by DY and YB—a square being described on BY, and a (rectangular) parallelogram (with short side equal to BY) completed on YD—the (rectangle contained) by DB and BY is thus also less than double the (rectangle contained) by DY and YB. And, KD being joined, the (rectangle contained) by DB and BY is equal to the (square) on BK, and the (rectangle contained) by DY and YB equal to the (square) on KY [Props. 3.31, 6.8 corr.]. Thus, the (square) on KB is less than double the (square) on KY. But, the (square) on KB is greater than double the (square) on BX. Thus, the (square) on KY (is) greater than the (square) on BX. And since BA is equal to KA, the (square) on BA is equal to the (square) on AK. And the (sum of the squares) on BX and XA is equal to the (square) on BA, and the (sum of the squares) on KY and YA (is) equal to the (square) on KA [Prop. 1.47]. Thus, the (sum of the squares) on BX and XA is equal to the (sum of the squares) on KY and YA, of which the (square) on KY (is) greater than the (square) on BX. Thus, the remaining (square) on YA is less than the (square) on XA. Thus, AX (is) greater than AY. Thus, AX is much greater than AG.[§] And AX is (a perpendicular) on one of the bases of the polyhedron, and AG (is a perpendicular) on the surface of the lesser sphere. Hence, the polyhedron will not touch the lesser sphere on its surface.

Thus, there being two spheres about the same center, a polyhedral solid has been inscribed in the greater sphere which does not touch the lesser sphere on its surface. (Which is) the very thing it was required to do.

Corollary

And, also, if a similar polyhedral solid to that in sphere $BCDE$ is inscribed in another sphere then the polyhedral solid in sphere $BCDE$ has to the polyhedral solid in the other sphere the cubed ratio that the diameter of sphere $BCDE$ has to the diameter of the other sphere. For if the solids are divided into similarly numbered, and similarly situated, pyramids, then the pyramids will be similar. And similar pyramids are in the cubed ratio of corresponding sides [Prop. 12.8 corr.]. Thus, the pyramid whose base is quadrilateral $KBPS$, and apex the point A, will have to the similarly situated pyramid in the other sphere the cubed ratio that a corresponding side (has) to a corresponding side. That is to say, that of radius AB of the sphere about center A to the radius of the other sphere. And, similarly, each pyramid in the sphere about center A will have to each similarly situated pyramid in the other sphere the cubed ratio that AB (has) to the radius of the other sphere. And as one of the leading (magnitudes is) to one of the following (in two sets of proportional magnitudes), so (the sum of) all the leading (magnitudes is) to (the sum of) all of the following (magnitudes) [Prop. 5.12]. Hence, the whole polyhedral solid in the sphere about center A will have to the whole polyhedral solid in the other [sphere] the cubed ratio that (radius) AB (has) to the radius of the other sphere. That is to say, that diameter BD (has) to the diameter of the other sphere. (Which is) the very thing it was required to show.