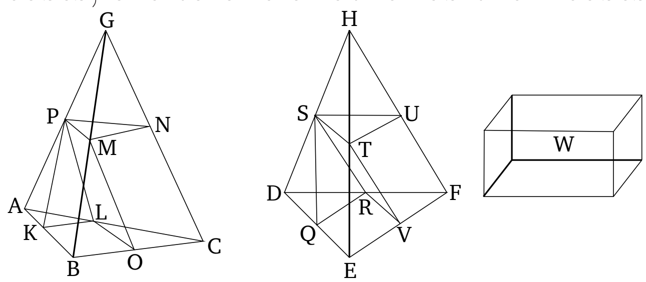
Book 12
Proposition 5

Pyramids which are of the same height, and have triangular bases, are to one another as their bases.



Let there be pyramids of the same height whose bases (are) the triangles ABC and DEF, and apexes the points G and H (respectively). I say that as base ABC is to base DEF, so pyramid $ABCG$ (is) to pyramid $DEFH$.

For if base ABC is not to base DEF, as pyramid $ABCG$ (is) to pyramid $DEFH$, then base ABC will be to base DEF, as pyramid $ABCG$ (is) to some solid either less than, or greater than, pyramid $DEFH$. Let it, first of all, be (in this ratio) to (some) lesser (solid), W. And let pyramid $DEFH$ have been divided into two pyramids equal to one another, and similar to the whole, and into two equal prisms. So, the (sum of the) two prisms is greater than half of the whole pyramid [Prop. 12.3]. And, again, let the pyramids generated by the division have been similarly divided, and let this be done continually until some pyramids are left from pyramid $DEFH$ which (when added together) are less than the excess by which pyramid $DEFH$ exceeds the solid W [Prop. 10.1]. Let them have been left, and, for the sake of argument, let them be $DQRS$ and $STUH$.

Thus, the (sum of the) remaining prisms within pyramid $DEFH$ is greater than solid W. Let pyramid $ABCG$ also have been divided similarly, and a similar number of times, as pyramid $DEFH$. Thus, as base ABC is to base DEF, so the (sum of the) prisms within pyramid $ABCG$ (is) to the (sum of the) prisms within pyramid $DEFH$ [Prop. 12.4]. But, also, as base ABC (is) to base DEF, so pyramid $ABCG$ (is) to solid W. And, thus, as pyramid $ABCG$ (is) to solid W, so the (sum of the) prisms within pyramid $ABCG$ (is) to the (sum of the) prisms within pyramid $DEFH$ [Prop. 5.11]. Thus, alternately, as pyramid $ABCG$ (is) to the (sum of the) prisms within it, so solid W (is) to the (sum of the) prisms within pyramid $DEFH$ [Prop. 5.16]. And pyramid $ABCG$ (is) greater than the (sum of the) prisms within it. Thus, solid W (is) also greater than the (sum of the) prisms within pyramid $DEFH$ [Prop. 5.14]. But, (it is) also less. This very thing is impossible. Thus, as base ABC is to base DEF, so pyramid $ABCG$ (is) not to some solid less than pyramid $DEFH$. So, similarly, we can show that base DEF is not to base ABC, as pyramid $DEFH$ (is) to some solid less than pyramid $ABCG$ either.

So, I say that neither is base ABC to base DEF, as pyramid $ABCG$ (is) to some solid greater than pyramid $DEFH$.

For, if possible, let it be (in this ratio) to some greater (solid), W. Thus, inversely, as base DEF (is) to base ABC, so solid W (is) to pyramid $ABCG$ [Prop. 5.7. corr.]. And as solid W (is) to pyramid $ABCG$, so pyramid $DEFH$ (is) to some (solid) less than pyramid $ABCG$, as shown before [Prop. 12.2 lem.]. And, thus, as base

DEF (is) to base ABC, so pyramid $DEFH$ (is) to some (solid) less than pyramid $ABCG$ [Prop. 5.11]. The very thing was shown (to be) absurd. Thus, base ABC is not to base DEF, as pyramid $ABCG$ (is) to some solid greater than pyramid $DEFH$. And, it was shown that neither (is it in this ratio) to a lesser (solid). Thus, as base ABC is to base DEF, so pyramid $ABCG$ (is) to pyramid $DEFH$. (Which is) the very thing it was required to show.