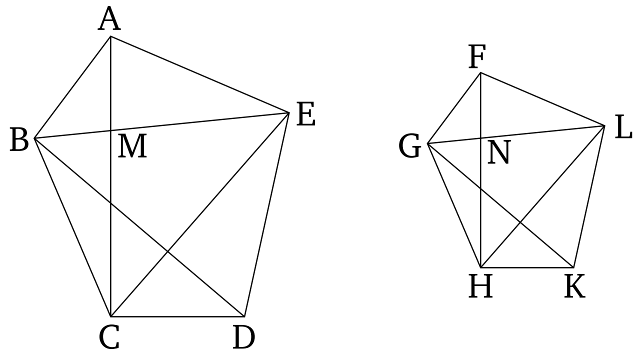
Book 6
Proposition 20

Similar polygons can be divided into equal numbers of similar triangles corresponding (in proportion) to the wholes, and one polygon has to the (other) polygon a squared ratio with respect to (that) a corresponding side (has) to a corresponding side.



Let $ABCDE$ and $FGHKL$ be similar polygons, and let AB correspond to FG. I say that polygons $ABCDE$ and $FGHKL$ can be divided into equal numbers of similar triangles corresponding (in proportion) to the wholes, and (that) polygon $ABCDE$ has a squared ratio to polygon $FGHKL$ with respect to that AB (has) to FG.

Let BE, EC, GL, and LH have been joined.

And since polygon $ABCDE$ is similar to polygon $FGHKL$, angle BAE is equal to angle GFL, and as BA is to AE, so GF (is) to FL [Def. 6.1]. Therefore, since ABE and FGL are two triangles having one angle equal to one angle and the sides about the equal angles proportional, triangle ABE is thus equiangular to triangle FGL [Prop. 6.6]. Hence, (they are) also similar [Prop. 6.4, Def. 6.1]. Thus, angle ABE is equal to (angle) FGL.

And the whole (angle) ABC is equal to the whole (angle) FGH, on account of the similarity of the polygons. Thus, the remaining angle EBC is equal to LGH. And since, on account of the similarity of triangles ABE and FGL, as EB is to BA, so LG (is) to GF, but also, on account of the similarity of the polygons, as AB is to BC, so FG (is) to GH, thus, via equality, as EB is to BC, so LG (is) to GH [Prop. 5.22], and the sides about the equal angles, EBC and LGH, are proportional. Thus, triangle EBC is equiangular to triangle LGH [Prop. 6.6]. Hence, triangle EBC is also similar to triangle LGH [Prop. 6.4, Def. 6.1]. So, for the same (reasons), triangle ECD is also similar to triangle LHK. Thus, the similar polygons $ABCDE$ and $FGHKL$ have been divided into equal numbers of similar triangles.

I also say that (the triangles) correspond (in proportion) to the wholes. That is to say, the triangles are proportional: ABE, EBC, and ECD are the leading (magnitudes), and their (associated) following (magnitudes are) FGL, LGH, and LHK (respectively). (I) also (say) that polygon $ABCDE$ has a squared ratio to polygon $FGHKL$ with respect to (that) a corresponding side (has) to a corresponding side—that is to say, (side) AB to FG.

For let AC and FH have been joined. And since angle ABC is equal to FGH, and as AB is to BC, so FG (is) to GH, on account of the similarity of the polygons, triangle ABC is equiangular to triangle FGH [Prop. 6.6]. Thus, angle BAC is equal to GFH, and (angle) BCA to GHF. And since angle BAM is equal to GFN, and (angle) ABM is also equal to FGN (see earlier), the remaining (angle) AMB is thus also equal to the remaining (angle) FNG [Prop. 1.32]. Thus, triangle ABM is equiangular to triangle FGN. So, similarly, we can show that triangle BMC is also equiangular to triangle GNH. Thus, proportionally, as AM is to MB, so FN (is) to NG, and as BM (is) to MC, so GN (is) to NH [Prop. 6.4]. Hence, also, via equality, as AM (is) to MC, so FN (is) to NH [Prop. 5.22]. But, as AM (is) to MC, so [triangle] ABM is to MBC, and AME to EMC. For they are to one another as their bases [Prop. 6.1]. And as one of the leading (magnitudes) is to one of the following (magnitudes), so (the sum of) all the leading (magnitudes) is to (the sum of) all the following (magnitudes) [Prop. 5.12]. Thus, as triangle AMB (is) to BMC, so (triangle) ABE (is) to CBE. But, as (triangle) AMB (is) to BMC, so AM (is) to MC. Thus, also, as AM (is) to MC, so triangle ABE (is) to triangle EBC. And so, for the same (reasons), as FN (is) to NH, so triangle FGL (is) to triangle GLH. And as AM is to MC, so FN (is) to NH. Thus, also, as triangle ABE (is) to triangle BEC, so triangle FGL (is) to triangle GLH, and, alternately, as triangle ABE (is) to triangle FGL, so triangle BEC (is) to triangle GLH [Prop. 5.16]. So, similarly, we can also show, by joining BD and GK, that as triangle BEC (is) to triangle LGH, so triangle ECD (is) to triangle LHK. And since as triangle ABE is to triangle FGL, so (triangle) EBC (is) to LGH, and, further, (triangle) ECD to LHK, and also as one of the leading (magnitudes is) to one of the following, so (the sum of) all the leading (magnitudes is) to (the sum of) all the following [Prop. 5.12], thus as triangle ABE is to triangle FGL, so polygon $ABCDE$ (is) to polygon $FGHKL$. But, triangle ABE has a squared ratio to triangle FGL with respect to (that) the corresponding side AB (has) to the corresponding side FG. For, similar triangles are in the squared ratio of corresponding sides [Prop. 6.14]. Thus, polygon $ABCDE$ also has a squared ratio to polygon $FGHKL$ with respect to (that) the corresponding side AB (has) to the corresponding side FG.

Thus, similar polygons can be divided into equal numbers of similar triangles corresponding (in proportion) to the wholes, and one polygon has to the (other) polygon a squared ratio with respect to (that) a corresponding side (has) to a corresponding side. [(Which is) the very thing it was required to show].

Corollary

And, in the same manner, it can also be shown for [similar] quadrilaterals that they are in the squared ratio of (their) corresponding sides. And it was also shown for triangles. Hence, in general, similar rectilinear figures are also to one another in the squared ratio of (their) corresponding sides. (Which is) the very thing it was required to show.