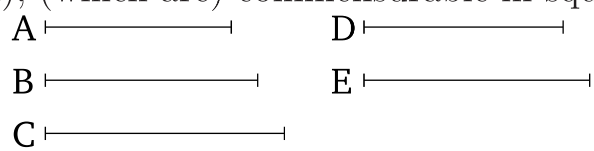
Book 10
Proposition 28

To find (two) medial (straight-lines), containing a medial (area), (which are) commensurable in square only.



Let the [three] rational (straight-lines) A, B, and C, (which are) commensurable in square only, be laid down. And let, D, the mean proportional (straight-line) to A and B, have been taken [Prop. 6.13]. And let it be contrived that as B (is) to C, (so) D (is) to E [Prop. 6.12].

Since the rational (straight-lines) A and B are commensurable in square only, the (rectangle contained) by A and B—that is to say, the (square) on D [Prop. 6.17]—is medial [Prop. 10.21]. Thus, D (is) medial [Prop. 10.21]. And since B and C are commensurable in square only, and as B is to C, (so) D (is) to E, D and E are thus commensurable in square only [Prop. 10.11]. And D (is) medial. E (is) thus also medial [Prop. 10.23]. Thus, D and E are medial (straight-lines which are) commensurable in square only. So, I say that they also enclose a medial (area). For since as B is to C, (so) D (is) to E, thus, alternately, as B (is) to D, (so) C (is) to E [Prop. 5.16]. And as B (is) to D, (so) D (is) to A. And thus as D (is) to A, (so) C (is) to E. Thus, the (rectangle contained) by A and C is equal to the (rectangle contained) by D and E [Prop. 6.16]. And the (rectangle contained) by A and C is medial [Prop. 10.21]. Thus, the (rectangle contained) by D and E (is) also medial.

Thus, (two) medial (straight-lines, D and E), containing a medial (area), (which are) commensurable in square only, have been found. (Which is) the very thing it was required to show.

Lemma

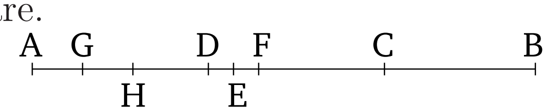
To find two square numbers such that the sum of them is also square.

Let the two numbers AB and BC be laid down. And let them be either (both) even or (both) odd. And since, if an even (number) is subtracted from an even (number), or if an odd (number is subtracted) from an odd (number), then the remainder is even [Props. 9.24, 9.26], the remainder AC is thus even. Let AC have been cut in half at D. And let AB and BC also be either similar plane (numbers), or square (numbers)—which are themselves also similar plane (numbers). Thus, the (number created) from (multiplying) AB and BC, plus the square on CD, is equal to the square on BD [Prop. 2.6]. And the (number created) from (multiplying) AB and BC is square—inasmuch as it was shown that if two similar plane (numbers) make some (number) by multiplying one another then the (number so) created is square [Prop. 9.1]. Thus, two square numbers have been found—(namely,) the (number created) from (multiplying) AB and BC, and the (square) on CD—which, (when) added (together), make the square on BD.

And (it is) clear that two square (numbers) have again been found—(namely,) the (square) on BD, and the (square) on CD—such that their difference—(namely,) the (rectangle) contained by AB and BC—is square whenever AB and BC are similar plane (numbers). But, when they are not similar plane numbers, two square (numbers) have been found—(namely,) the (square) on BD, and the (square) on DC—between which the difference—(namely,) the (rectangle) contained by AB and BC—is not square. (Which is) the very thing it was required to show.

Lemma II

To find two square numbers such that the sum of them is not square.



For let the (number created) from (multiplying) AB and BC, as we said, be square. And (let) CA (be) even. And let CA have been cut in half at D. So it is clear that the square (number created) from (multiplying) AB and BC, plus the square on CD, is equal to the square on BD [see previous lemma]. Let the unit DE have been subtracted (from BD). Thus, the (number created) from (multiplying) AB and BC, plus the (square) on CE, is less than the square on BD. I say, therefore, that the square (number created) from (multiplying) AB and BC, plus the (square) on CE, is not square.

For if it is square, it is either equal to the (square) on BE, or less than the (square) on BE, but cannot any more be greater (than the square on BE), lest the unit be divided. First of all, if possible, let the (number created) from (multiplying) AB and BC, plus the (square) on CE, be equal to the (square) on BE. And let GA be double the unit DE. Therefore, since the whole of AC is double the whole of CD, of which AG is double DE, the remainder GC is thus double the remainder EC. Thus, GC has been cut in half at E. Thus, the (number created) from (multiplying) GB and BC, plus the (square) on CE, is equal to the square on BE [Prop. 2.6]. But, the (number created) from (multiplying) AB and BC, plus the (square) on CE, was also assumed (to be) equal to the square on BE. Thus, the (number created) from (multiplying) GB and BC, plus the (square) on CE, is equal to the (number created) from (multiplying) AB and BC, plus the (square) on CE. And subtracting the (square) on CE from both, AB is inferred (to be) equal to GB. The very thing is absurd. Thus, the (number created) from (multiplying) AB and BC, plus the (square) on CE, is not equal to the (square) on BE. So I say that (it is) not less than the (square) on BE either. For, if possible, let it be equal to the (square) on BF. And (let) HA (be) double DF. And it can again be inferred that HC (is) double CF. Hence, CH has also been cut in half at F. And, on account of this, the (number created) from (multiplying) HB and BC, plus the (square) on FC, becomes equal to the (square) on BF [Prop. 2.6]. And the (number created) from (multiplying) AB and BC, plus the (square) on CE, was also assumed (to be) equal to the (square) on BF. Hence, the (number created) from (multiplying) HB and BC, plus the (square) on CF, will also be equal to the (number created) from (multiplying)

AB and BC, plus the (square) on CE. The very thing is absurd. Thus, the (number created) from (multiplying) AB and BC, plus the (square) on CE, is not equal to less than the (square) on BE. And it was shown that (is it) not equal to the (square) on BE either. Thus, the (number created) from (multiplying) AB and BC, plus the square on CE, is not square. (Which is) the very thing it was required to show.