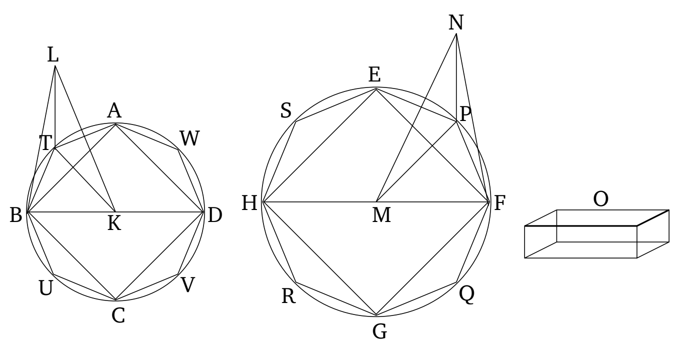
Book 12
Proposition 12

Similar cones and cylinders are to one another in the cubed ratio of the diameters of their bases.

Let there be similar cones and cylinders of which the bases (are) the circles $ABCD$ and $EFGH$, the diameters of the bases (are) BD and FH, and the axes of the cones and cylinders (are) KL and MN (respectively). I say that the cone whose base [is] circle $ABCD$, and apex the point L, has to the cone whose base [is] circle $EFGH$, and apex the point N, the cubed ratio that BD (has) to FH.



For if cone $ABCDL$ does not have to cone $EFGHN$ the cubed ratio that BD (has) to FH then cone $ABCDL$ will have the cubed ratio to some solid either less than, or greater than, cone $EFGHN$. Let it, first of all, have (such a ratio) to (some) lesser (solid), O. And let the square $EFGH$ have been inscribed in circle $EFGH$ [Prop. 4.6]. Thus, square $EFGH$ is greater than half of circle $EFGH$ [Prop. 12.2]. And let a pyramid having the same apex as the cone have been set up on square $EFGH$. Thus, the pyramid set up is greater than the half part of the cone

[Prop. 12.10]. So, let the circumferences EF, FG, GH, and HE have been cut in half at points P, Q, R, and S (respectively). And let EP, PF, FQ, QG, GR, RH, HS, and SE have been joined. And, thus, each of the triangles EPF, FQG, GRH, and HSE is greater than the half part of the segment of circle $EFGH$ about it [Prop. 12.2]. And let a pyramid having the same apex as the cone have been set up on each of the triangles EPF, FQG, GRH, and HSE. And thus each of the pyramids set up is greater than the half part of the segment of the cone about it [Prop. 12.10]. So, (if) the the remaining circumferences are cut in half, and straight-lines are joined, and pyramids having the same apex as the cone are set up on each of the triangles, and this is done continually, then we will (eventually) leave some segments of the cone whose (sum) is less than the excess by which cone $EFGHN$ exceeds solid O [Prop. 10.1]. Let them have been left, and let them be the (segments) on EP, PF, FQ, QG, GR, RH, HS, and SE. Thus, the remaining pyramid whose base is polygon $EPFQGRHS$, and apex the point N, is greater than solid O. And let the polygon $ATBUCVDW$, similar, and similarly laid out, to polygon $EPFQGRHS$, have been inscribed in circle $ABCD$ [Prop. 6.18]. And let a pyramid having the same apex as the cone have been set up on polygon $ATBUCVDW$. And let LBT be one of the triangles containing the pyramid whose base is polygon $ATBUCVDW$, and apex the point L. And let NFP be one of the triangles containing the pyramid whose base is triangle $EPFQGRHS$, and apex the point N.

And let KT and MP have been joined. And since cone $ABCDL$ is similar to cone $EFGHN$, thus as BD is to FH, so axis KL (is) to axis MN [Def. 11.24]. And as BD (is) to FH, so BK (is) to FM. And, thus, as BK (is) to FM, so KL (is) to MN. And, alternately, as BK (is) to KL, so FM (is) to MN [Prop. 5.16]. And the sides around the equal angles BKL and FMN are proportional. Thus, triangle BKL is similar to triangle FMN [Prop. 6.6]. Again, since as BK (is) to KT, so FM (is) to MP, and (they are) about the equal angles BKT and FMP, inasmuch as whatever part angle BKT is of the four right-angles at the center K, angle FMP is also the same part of the four right-angles at the center M. Therefore, since the sides about equal angles are proportional, triangle BKT is thus similar to traingle FMP [Prop. 6.6]. Again, since it was shown that as BK (is) to KL, so FM (is) to MN, and BK (is) equal to KT, and FM to PM, thus as TK (is) to KL, so PM (is) to MN. And the sides about the equal angles TKL and PMN—for (they are both) right-angles—are proportional. Thus, triangle LKT (is) similar to triangle NMP [Prop. 6.6]. And since, on account of the similarity of triangles LKB and NMF, as LB (is) to BK, so NF (is) to FM, and, on account of the similarity of triangles BKT and FMP, as KB (is) to BT, so MF (is) to FP [Def. 6.1], thus, via equality, as LB (is) to BT, so NF (is) to FP [Prop. 5.22]. Again, since, on account of the similarity of triangles LTK and NPM, as LT (is) to TK, so NP (is) to PM, and, on account of the similarity of triangles TKB and PMF, as KT (is) to TB, so MP

(is) to PF, thus, via equality, as LT (is) to TB, so NP (is) to PF [Prop. 5.22]. And it was shown that as TB (is) to BL, so PF (is) to FN. Thus, via equality, as TL (is) to LB, so PN (is) to NF [Prop. 5.22]. Thus, the sides of triangles LTB and NPF are proportional. Thus, triangles LTB and NPF are equiangular [Prop. 6.5]. And, hence, (they are) similar [Def. 6.1]. And, thus, the pyramid whose base is triangle BKT, and apex the point L, is similar to the pyramid whose base is triangle FMP, and apex the point N. For they are contained by equal numbers of similar planes [Def. 11.9]. And similar pyramids which also have triangular bases are in the cubed ratio of corresponding sides [Prop. 12.8]. Thus, pyramid $BKTL$ has to pyramid $FMPN$ the cubed ratio that BK (has) to FM. So, similarly, joining straight-lines from (points) A, W, D, V, C, and U to (center) K, and from (points) E, S, H, R, G, and Q to (center) M, and setting up pyramids having the same apexes as the cones on each of the triangles (so formed), we can also show that each of the pyramids (on base $ABCD$ taken) in order will have to each of the pyramids (on base $EFGH$ taken) in order the cubed ratio that the corresponding side BK (has) to the corresponding side FM—that is to say, that BD (has) to FH. And (for two sets of proportional magnitudes) as one of the leading (magnitudes is) to one of the following, so (the sum of) all of the leading (magnitudes is) to (the sum of) all of the following (magnitudes) [Prop. 5.12]. And, thus, as pyramid $BKTL$ (is) to pyramid $FMPN$, so the whole pyramid whose base is polygon $ATBUCVDW$, and apex the point L, (is) to the whole pyramid whose base is polygon $EPFQGRHS$, and apex the point N. And, hence, the pyramid whose base is polygon $ATBUCVDW$, and apex the point L, has to the pyramid whose base is polygon $EPFQGRHS$, and apex the point N, the cubed ratio that BD (has) to FH. And it was also assumed that the cone whose base is circle $ABCD$, and apex the point L, has to solid O the cubed ratio that BD (has) to FH. Thus, as the cone whose base is circle $ABCD$, and apex the point L, is to solid O, so the pyramid whose base (is) [polygon] $ATBUCVDW$, and apex the point L, (is) to the pyramid whose base is polygon $EPFQGRHS$, and apex the point N. Thus, alternately, as the cone whose base (is) circle $ABCD$, and apex the point L, (is) to the pyramid within it whose base (is) the polygon $ATBUCVDW$, and apex the point L, so the [solid] O (is) to the pyramid whose base is polygon $EPFQGRHS$, and apex the point N [Prop. 5.16]. And the aforementioned cone (is) greater than the pyramid within it. For it encompasses it. Thus, solid O (is) also greater than the pyramid whose base is polygon $EPFQGRHS$, and apex the point N. But, (it is) also less. The very thing is impossible. Thus, the cone whose base (is) circle $ABCD$, and apex the [point] L, does not have to some solid less than the cone whose base (is) circle $EFGH$, and apex the point N, the cubed ratio that BD (has) to EH. So, similarly, we can show that neither does cone $EFGHN$ have to some solid less than cone $ABCDL$ the cubed ratio that FH (has) to BD.

So, I say that neither does cone $ABCDL$ have to some solid greater than cone $EFGHN$ the cubed ratio that BD (has) to FH.

For, if possible, let it have (such a ratio) to a greater (solid), O. Thus, inversely, solid O has to cone $ABCDL$ the cubed ratio that FH (has) to BD [Prop. 5.7 corr.]. And as solid O (is) to cone $ABCDL$, so cone $EFGHN$ (is) to some solid less than cone $ABCDL$ [12.2 lem.]. Thus, cone $EFGHN$ also has to some solid less than cone $ABCDL$ the cubed ratio that FH (has) to BD. The very thing was shown (to be) impossible. Thus, cone $ABCDL$ does not have to some solid greater than cone $EFGHN$ the cubed ratio than BD (has) to FH. And it was shown that neither (does it have such a ratio) to a lesser (solid). Thus, cone $ABCDL$ has to cone $EFGHN$ the cubed ratio that BD (has) to FG.

And as the cone (is) to the cone, so the cylinder (is) to the cylinder. For a cylinder is three times a cone on the same base as the cone, and of the same height as it [Prop. 12.10]. Thus, the cylinder also has to the cylinder the cubed ratio that BD (has) to FH.

Thus, similar cones and cylinders are in the cubed ratio of the diameters of their bases. (Which is) the very thing it was required to show.