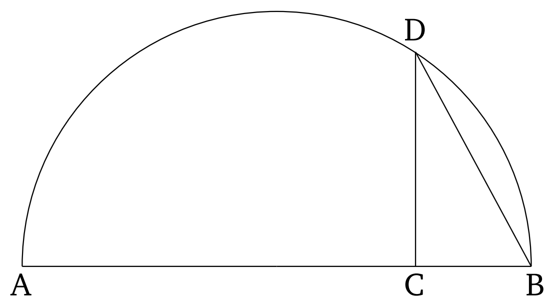
Book 13
Proposition 16

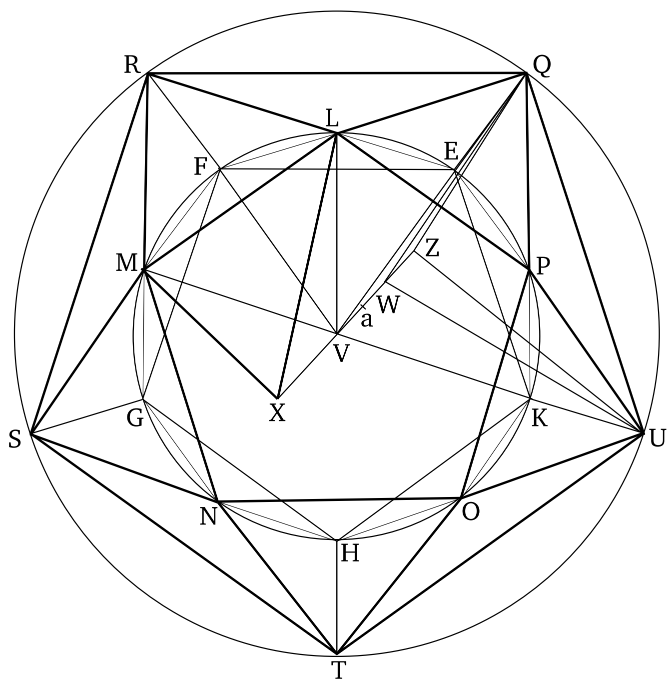
To construct an icosahedron, and to enclose (it) in a sphere, like the aforementioned figures, and to show that the side of the icosahedron is that irrational (straight-line) called minor.



Let the diameter AB of the given sphere be laid out, and let it have been cut at C such that AC is four times CB [Prop. 6.10]. And let the semi-circle ADB have been drawn on AB. And let the straight-line CD have been drawn from C at right-angles to AB. And let DB have been joined. And let the circle $EFGHK$ be set down, and let its radius be equal to DB. And let the equilateral and equiangular pentagon $EFGHK$ have been inscribed in circle $EFGHK$ [Prop. 4.11]. And let the circumferences EF, FG, GH, HK, and KE have been cut in half at points L, M, N, O, and P (respectively). And let LM, MN, NO, OP, PL, and EP have been joined. Thus, pentagon $LMNOP$ is also equilateral, and EP (is) the side of the decagon (inscribed in the circle). And let the straight-lines EQ, FR, GS, HT, and KU, which are equal to the radius of circle $EFGHK$, have been set up at right-angles to the plane of the circle, at points E, F, G, H, and K (respectively). And let

QR, RS, ST, TU, UQ, QL, LR, RM, MS, SN, NT, TO, OU, UP, and PQ have been joined.

And since EQ and KU are each at right-angles to the same plane, EQ is thus parallel to KU [Prop. 11.6]. And it is also equal to it. And straight-lines joining equal and parallel (straight-lines) on the same side are (themselves) equal and parallel [Prop. 1.33]. Thus, QU is equal and parallel to EK. And EK (is the side) of an equilateral pentagon (inscribed in circle $EFGHK$). Thus, QU (is) also the side of an equilateral pentagon inscribed in circle $EFGHK$. So, for the same (reasons), QR, RS, ST, and TU are also the sides of an equilateral pentagon inscribed in circle $EFGHK$. Pentagon $QRSTU$ (is) thus equilateral. And side QE is (the side) of a hexagon (inscribed in circle $EFGHK$), and EP (the side) of a decagon, and (angle) QEP is a right-angle, thus QP is (the side) of a pentagon (inscribed in the same circle). For the square on the side of a pentagon is (equal to the sum of) the (squares) on (the sides of) a hexagon and a decagon inscribed in the same circle [Prop. 13.10]. So, for the same (reasons), PU is also the side of a pentagon. And QU is also (the side) of a pentagon. Thus, triangle QPU is equilateral. So, for the same (reasons), (triangles) QLR, RMS, SNT, and TOU are each also equilateral. And since QL and QP were each shown (to be the sides) of a pentagon, and LP is also (the side) of a pentagon, triangle QLP is thus equilateral. So, for the same (reasons), triangles LRM, MSN, NTO, and OUP are each also equilateral.



Let the center, point V, of circle $EFGHK$ have been found [Prop. 3.1]. And let VZ have been set up, at (point) V, at right-angles to the plane of the circle. And let it have been produced on the other side (of the circle), like VX. And let VW have been cut off (from XZ so as to be equal to the side) of a hexagon, and each of VX and WZ (so as to be equal to the side) of a decagon. And let QZ, QW, UZ, EV, LV, LX, and XM have been joined.

And since VW and QE are each at right-angles to the plane of the circle, VW is thus parallel to QE [Prop. 11.6]. And they are also equal. EV and QW are thus equal and parallel (to one another) [Prop. 1.33]. And EV (is the side) of a hexagon. Thus, QW (is) also (the side) of a hexagon. And since QW is (the side) of a hexagon, and WZ (the side) of a decagon, and angle QWZ is a right-angle [Def. 11.3, Prop. 1.29], QZ is thus (the side) of a pentagon [Prop. 13.10]. So, for the same (reasons), UZ is also (the side) of a pentagon—inasmuch as, if we join VK and WU then they will be equal and opposite. And VK, being (equal) to the radius (of the circle), is (the side) of a hexagon [Prop. 4.15 corr.]. Thus, WU (is) also the side of a hexagon. And WZ (is the side) of a decagon, and (angle) UWZ (is) a right-angle. Thus, UZ (is the side) of a pentagon [Prop. 13.10]. And QU is also (the side) of a pentagon. Triangle QUZ is thus equilateral. So, for the same (reasons), each of the remaining triangles, whose bases are the straight-lines QR, RS, ST, and TU, and apexes the point Z, are also equilateral. Again, since VL (is the side) of a hexagon, and VX (the side) of a decagon, and angle LVX is a right-angle, LX is thus (the side) of a pentagon [Prop. 13.10]. So, for the same (reasons), if we join MV, which is (the side) of a hexagon, MX is also inferred (to be the side) of a pentagon. And LM is also (the side) of a pentagon. Thus, triangle LMX is equilateral. So, similarly, it can be shown that each of the remaining triangles, whose bases are the (straight-lines) MN, NO, OP, and PL, and apexes the point X, are also equilateral. Thus, an icosahedron contained by twenty equilateral triangles has been constructed.

So, it is also necessary to enclose it in the given sphere, and to show that the side of the icosahedron is that irrational (straight-line) called minor.

For, since VW is (the side) of a hexagon, and WZ (the side) of a decagon, VZ has thus been cut in extreme and mean ratio at W, and VW is its greater piece [Prop. 13.9]. Thus, as ZV is to VW, so VW (is) to WZ. And VW (is) equal to VE, and WZ to VX. Thus, as ZV is to VE, so EV (is) to VX. And angles ZVE and EVX are right-angles. Thus, if we join straight-line EZ then angle XEZ will be a right-angle, on account of the similarity of triangles XEZ and VEZ. [Prop. 6.8]. So, for the same (reasons), since as ZV is to VW, so VW (is) to WZ, and ZV (is) equal to XW, and VW to WQ, thus as XW is to WQ, so QW (is) to WZ. And, again, on account of this, if we join QX then the angle at Q will be a right-angle [Prop. 6.8]. Thus, the semi-circle drawn on XZ will also pass through Q [Prop. 3.31]. And if XZ remains fixed, and the semi-circle is carried around, and again established at the same (position) from which it began to be moved, then it will also pass through (point) Q, and (through) the remaining (angular) points of the icosahedron. And the icosahedron will have been enclosed by a sphere. So, I say that (it is) also (enclosed) by the given (sphere). For let VW have been cut in half at a. And since the straight-line VZ has been cut in extreme and mean ratio at W, and ZW is its lesser piece, then the square on ZW added to half of the greater piece, Wa, is five times the (square) on half of the greater piece [Prop. 13.3]. Thus, the (square) on Za is five times the (square) on aW. And ZX is double Za, and VW double aW. Thus, the (square) on ZX is five times the (square) on WV. And since AC is four times CB, AB is thus five times BC. And as AB (is) to BC, so the (square) on AB (is) to the (square) on BD

[Prop. 6.8, Def. 5.9]. Thus, the (square) on AB is five times the (square) on BD. And the (square) on ZX was also shown (to be) five times the (square) on VW. And DB is equal to VW. For each of them is equal to the radius of circle $EFGHK$. Thus, AB (is) also equal to XZ. And AB is the diameter of the given sphere. Thus, XZ is equal to the diameter of the given sphere. Thus, the icosahedron has been enclosed by the given sphere.

So, I say that the side of the icosahedron is that irrational (straight-line) called minor. For since the diameter of the sphere is rational, and the square on it is five times the (square) on the radius of circle $EFGHK$, the radius of circle $EFGHK$ is thus also rational. Hence, its diameter is also rational. And if an equilateral pentagon is inscribed in a circle having a rational diameter then the side of the pentagon is that irrational (straight-line) called minor [Prop. 13.11]. And the side of pentagon $EFGHK$ is (the side) of the icosahedron. Thus, the side of the icosahedron is that irrational (straight-line) called minor.

Corollary

So, (it is) clear, from this, that the square on the diameter of the sphere is five times the square on the radius of the circle from which the icosahedron has been described, and that the the diameter of the sphere is the sum of (the side) of the hexagon, and two of (the sides) of the decagon, inscribed in the same circle.