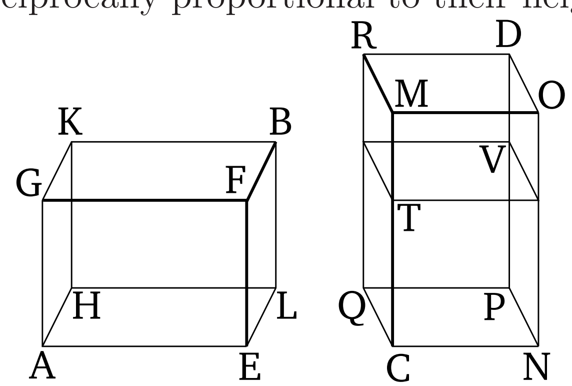
Book 11
Proposition 34

The bases of equal parallelepiped solids are reciprocally proportional to their heights. And those parallelepiped solids whose bases are reciprocally proportional to their heights are equal.

Let AB and CD be equal parallelepiped solids. I say that the bases of the parallelepiped solids AB and CD are reciprocally proportional to their heights, and (so) as base EH is to base NQ, so the height of solid CD (is) to the height of solid AB.

For, first of all, let the (straight-lines) standing up, AG, EF, LB, HK, CM, NO, PD, and QR, be at right-angles to their bases. I say that as base EH is to base NQ, so CM (is) to AG.

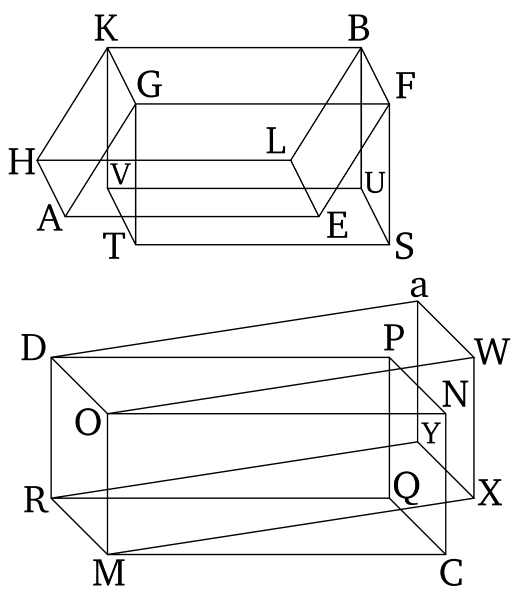
Therefore, if base EH is equal to base NQ, and solid AB is also equal to solid CD, CM will also be equal to AG. For parallelepiped solids of the same height are to one another as their bases [Prop. 11.32]. And as base EH (is) to NQ, so CM will be to AG. And (so it is) clear that the bases of the parallelepiped solids AB and CD are reciprocally proportional to their heights.



So let base EH not be equal to base NQ, but let EH be greater. And solid AB is also equal to solid CD. Thus, CM is also greater than AG. Therefore, let CT be made equal to AG. And let the parallelepiped solid VC have been completed on the base NQ, with height CT. And since solid AB is equal to solid CD, and CV (is) extrinsic (to them), and equal (magnitudes) have the same ratio to the same (magnitude) [Prop. 5.7], thus as solid AB is to solid CV, so solid CD (is) to solid CV. But, as solid AB (is) to solid CV, so base EH (is) to base NQ. For the solids AB and CV (are) of equal height [Prop. 11.32]. And as solid CD (is) to solid CV, so base MQ (is) to base TQ [Prop. 11.25], and CM to CT [Prop. 6.1]. And, thus, as base EH is to base NQ, so MC (is) to AG. And CT (is) equal to AG. And thus as base EH (is) to base NQ, so MC (is) to AG. Thus, the bases of the parallelepiped solids AB and CD are reciprocally proportional to their heights.

So, again, let the bases of the parallelepipid solids AB and CD be reciprocally proportional to their heights, and let base EH be to base NQ, as the height of solid CD (is) to the height of solid AB. I say that solid AB is equal to solid CD. [For] let the (straight-lines) standing up again be at right-angles to the bases. And if base EH is equal to base NQ, and as base EH is to base NQ, so the height of solid CD (is) to the height of solid AB, the height of solid CD is thus also equal to the height of solid AB. And parallelepiped solids on equal bases, and also with the same height, are equal to one another [Prop. 11.31]. Thus, solid AB is equal to solid CD.

So, let base EH not be equal to [base] NQ, but let EH be greater. Thus, the height of solid CD is also greater than the height of solid AB, that is to say CM (greater) than AG. Let CT again be made equal to AG, and let the solid CV have been similarly completed. Since as base EH is to base NQ, so MC (is) to AG, and AG (is) equal to CT, thus as base EH (is) to base NQ, so CM (is) to CT. But, as [base] EH (is) to base NQ, so solid AB (is) to solid CV. For solids AB and CV are of equal heights [Prop. 11.32]. And as CM (is) to CT, so (is) base MQ to base QT [Prop. 6.1], and solid CD to solid CV [Prop. 11.25]. And thus as solid AB (is) to solid CV, so solid CD (is) to solid CV. Thus, AB and CD each have the same ratio to CV. Thus, solid AB is equal to solid CD [Prop. 5.9].



So, let the (straight-lines) standing up, FE, BL, GA, KH, ON, DP, MC, and RQ, not be at right-angles to their bases. And let perpendiculars have been drawn to the planes through EH and NQ from points F, G, B, K, O, M, R, and D, and let them have joined the planes at (points) S, T, U, V, W, X, Y, and a (respectively). And let the solids FV and OY have been completed. In this case, also, I say that the solids AB and CD being equal, their bases are reciprocally proportional to their heights, and (so) as base EH is to base NQ, so the height of solid CD (is) to the height of solid AB.

Since solid AB is equal to solid CD, but AB is equal to BT. For they are on the same base FK, and (have) the same height [Props. 11.29, 11.30]. And solid CD is equal is equal to DX. For Solid BT is thus also equal to solid DX. Thus, as base FK (is) to base OR, so the height of solid DX (is) to the height of solid BT (see first part of proposition). And base FK (is) equal to base EH, and base OR to NQ. Thus, as base EH is to base NQ, so the height of solid DX (is) to the height of solid BT. And solids DX, BT are the same height as (solids) DC, BA (respectively). Thus, as base EH is to base NQ, so the height of solid DC (is) to the height of solid AB. Thus, the bases of the parallelepiped solids AB and CD are reciprocally proportional to their heights.

So, again, let the bases of the parallelepiped solids AB and CD be reciprocally proportional to their heights, and (so) let base EH be to base NQ, as the height of solid CD (is) to the height of solid AB. I say that solid AB is equal to solid CD.

For, with the same construction (as before), since as base EH is to base NQ, so the height of solid CD (is) to the height of solid AB, and base EH (is) equal to base

FK, and NQ to OR, thus as base FK is to base OR, so the height of solid CD (is) to the height of solid AB. And solids AB, CD are the same height as (solids) BT, DX (respectively). Thus, as base FK is to base OR, so the height of solid DX (is) to the height of solid BT. Thus, the bases of the parallelepiped solids BT and DX are reciprocally proportional to their heights. Thus, solid BT is equal to solid DX (see first part of proposition). But, BT is equal to BA. For [they are] on the same base FK, and (have) the same height [Props. 11.29, 11.30]. And solid DX is equal to solid DC [Props. 11.29, 11.30]. Thus, solid AB is also equal to solid CD. (Which is) the very thing it was required to show.