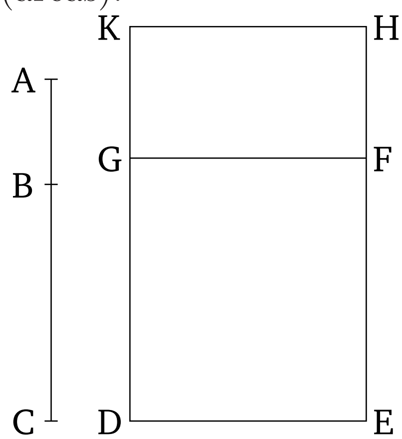
Book 10
Proposition 41

If two straight-lines (which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, moreover, incommensurable with the sum of the squares on them, are added together then the whole straight-line is irrational—let it be called the square-root of (the sum of) two medial (areas).



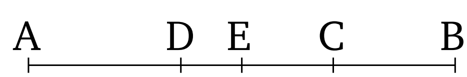
For let the two straight-lines, AB and BC, incommensurable in square, (and) fulfilling the prescribed (conditions), be laid down together [Prop. 10.35]. I say that AC is irrational.

Let the rational (straight-line) DE be laid out, and let (the rectangle) DF, equal to (the sum of) the (squares) on AB and BC, and (the rectangle) GH, equal to twice the (rectangle contained) by AB and BC, have been applied to DE. Thus, the whole of DH is equal to the square on AC [Prop. 2.4]. And since the sum of the (squares) on AB and BC is medial, and is equal to DF,

DF is thus also medial. And it is applied to the rational (straight-line) DE. Thus, DG is rational, and incommensurable in length with DE [Prop. 10.22]. So, for the same (reasons), GK is also rational, and incommensurable in length with GF—that is to say, DE. And since (the sum of) the (squares) on AB and BC is incommensurable with twice the (rectangle contained) by AB and BC, DF is incommensurable with GH. Hence, DG is also incommensurable (in length) with GK [Props. 6.1, 10.11]. And they are rational. Thus, DG and GK are rational (straight-lines which are) commensurable in square only. Thus, DK is irrational, and that (straight-line which is) called binomial [Prop. 10.36]. And DE (is) rational. Thus, DH is irrational, and its square-root is irrational [Def. 10.4]. And AC (is) the square-root of HD. Thus, AC is irrational—let it be called the square-root of (the sum of) two medial (areas). (Which is) the very thing it was required to show.

Lemma

We will now demonstrate that the aforementioned irrational (straight-lines) are uniquely divided into the straight-lines of which they are the sum, and which produce the prescribed types, (after) setting forth the following lemma.



Let the straight-line AB be laid out, and let the whole (straight-line) have been cut into unequal parts at each of the (points) C and D. And let AC be assumed (to be) greater than DB. I say that (the sum of) the (squares) on AC and CB is greater than (the sum of) the (squares) on AD and DB.

For let AB have been cut in half at E. And since AC is greater than DB, let DC have been subtracted from both. Thus, the remainder AD is greater than the remainder CB. And AE (is) equal to EB. Thus, DE (is) less than EC. Thus, points C and D are not equally far from the point of bisection. And since the (rectangle contained) by AC and CB, plus the (square) on EC, is equal to the (square) on EB [Prop. 2.5], but, moreover, the (rectangle contained) by AD and DB, plus the (square) on DE, is also equal to the (square) on EB [Prop. 2.5], the (rectangle contained) by AC and CB, plus the (square) on EC, is thus equal to the (rectangle contained) by AD and DB, plus the (square) on DE. And, of these, the (square) on DE is less than the (square) on EC. And, thus, the remaining (rectangle contained) by AC and CB is less than the (rectangle contained) by AD and DB. And, hence, twice the (rectangle contained) by AC and CB is less than twice the (rectangle contained) by AD and DB. And thus the remaining sum of the (squares) on AC and CB is greater than the sum of the (squares) on AD and DB. (Which is) the very thing it was required to show.