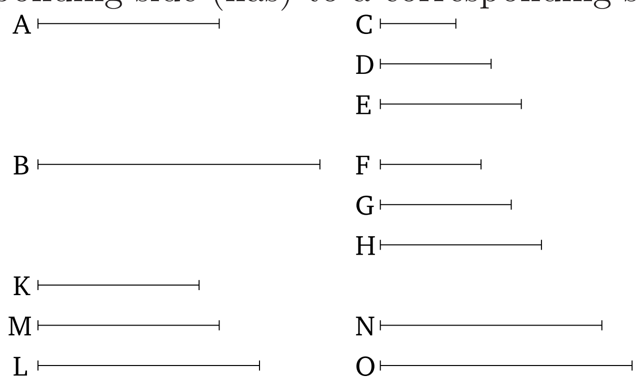
Book 8
Proposition 19

Two numbers fall (between) two similar solid numbers in mean proportion. And a solid (number) has to a similar solid (number) a cubed ratio with respect to (that) a corresponding side (has) to a corresponding side.



Let A and B be two similar solid numbers, and let C, D, E be the sides of A, and F, G, H (the sides) of B. And since similar solid (numbers) are those having proportional sides [Def. 7.21], thus as C is to D, so F (is) to G, and as D (is) to E, so G (is) to H. I say that two numbers fall (between) A and B in mean proportion, and (that) A has to B a cubed ratio with respect to (that) C (has) to F, and D to G, and, further, E to H.

For let C make K (by) multiplying D, and let F make L (by) multiplying G. And since C, D are in the same ratio as F, G, and K is the (number created) from (multiplying) C, D, and L the (number created) from (multiplying) F, G, [thus] K and L are similar plane numbers [Def. 7.21]. Thus, there exits one number in mean proportion to K and L [Prop. 8.18]. Let it be M. Thus,

M is the (number created) from (multiplying) D, F, as shown in the theorem before this (one). And since D has made K (by) multiplying C, and has made M (by) multiplying F, thus as C is to F, so K (is) to M [Prop. 7.17]. But, as K (is) to M, (so) M (is) to L. Thus, K, M, L are continuously proportional in the ratio of C to F. And since as C is to D, so F (is) to G, thus, alternately, as C is to F, so D (is) to G [Prop. 7.13]. And so, for the same (reasons), as D (is) to G, so E (is) to H. Thus, K, M, L are continuously proportional in the ratio of C to F, and of D to G, and, further, of E to H. So let E, H make N, O, respectively, (by) multiplying M. And since A is solid, and C, D, E are its sides, E has thus made A (by) multiplying the (number created) from (multiplying) C, D. And K is the (number created) from (multiplying) C, D. Thus, E has made A (by) multiplying K. And so, for the same (reasons), H has made B (by) multiplying L. And since E has made A (by) multiplying K, but has, in fact, also made N (by) multiplying M, thus as K is to M, so A (is) to N [Prop. 7.17]. And as K (is) to M, so C (is) to F, and D to G, and, further, E to H. And thus as C (is) to F, and D to G, and E to H, so A (is) to N. Again, since E, H have made N, O, respectively, (by) multiplying M, thus as E is to H, so N (is) to O [Prop. 7.18]. But, as E (is) to H, so C (is) to F, and D to G. And thus as C (is) to F, and D to G, and E to H, so (is) A to N, and N to O. Again, since H has made O (by) multiplying M, but has, in fact, also made B (by) multiplying L, thus as M (is) to L, so O (is) to B [Prop. 7.17]. But, as M (is) to L, so C (is) to F, and

D to G, and E to H. And thus as C (is) to F, and D to G, and E to H, so not only (is) O to B, but also A to N, and N to O. Thus, A, N, O, B are continuously proportional in the aforementioned ratios of the sides.

So I say that A also has to B a cubed ratio with respect to (that) a corresponding side (has) to a corresponding side—that is to say, with respect to (that) the number C (has) to F, or D to G, and, further, E to H. For since A, N, O, B are four continuously proportional numbers, A thus has to B a cubed ratio with respect to (that) A (has) to N [Def. 5.10]. But, as A (is) to N, so it was shown (is) C to F, and D to G, and, further, E to H. And thus A has to B a cubed ratio with respect to (that) a corresponding side (has) to a corresponding side—that is to say, with respect to (that) the number C (has) to F, and D to G, and, further, E to H. (Which is) the very thing it was required to show.