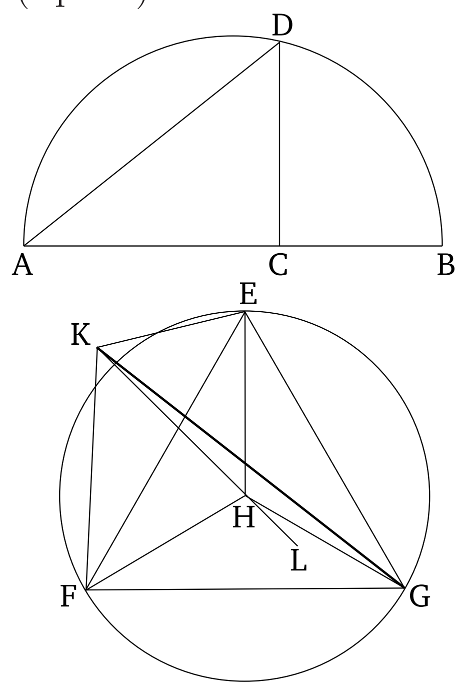
Book 13
Proposition 13

To construct a (regular) pyramid (*i.e.*, a tetrahedron), and to enclose (it) in a given sphere, and to show that the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.



Let the diameter AB of the given sphere be laid out, and let it have been cut at point C such that AC is double CB [Prop. 6.10]. And let the semi-circle ADB have been drawn on AB. And let CD have been drawn from point C at right-angles to AB. And let DA have been joined. And let the circle EFG be laid down having a radius equal to DC, and let the equilateral triangle EFG have been inscribed in circle EFG [Prop. 4.2]. And let the center of the circle, point H, have been found [Prop. 3.1]. And let EH, HF, and HG have been joined. And let HK have been set up, at point H, at right-angles to the plane of circle EFG [Prop. 11.12]. And let HK, equal to the straight-line AC, have been cut off from HK. And let KE, KF, and KG have been joined. And since KH is at right-angles to the plane of circle EFG, it will thus also make right-angles with all of the straight-lines joining it (which are) also in the plane of circle EFG [Def. 11.3]. And HE, HF, and HG each join it. Thus, HK is at right-angles to each of HE, HF, and HG. And since AC is equal to HK, and CD to HE, and they contain right-angles, the base DA is thus equal to the base KE [Prop. 1.4]. So, for the same (reasons), KF and KG is each equal to DA. Thus, the three (straight-lines) KE, KF, and KG are equal to one another. And since AC is double CB, AB (is) thus triple BC. And as AB (is) to BC, so the (square) on AD (is) to the (square) on DC, as will be shown later [see lemma]. Thus, the (square) on AD (is) three times the (square) on DC. And the (square) on FE is also three times the (square) on EH [Prop. 13.12], and DC is equal to EH. Thus, DA (is) also equal to EF. But, DA was shown (to be) equal to each of KE, KF, and KG. Thus, EF, FG, and GE are equal to KE, KF, and KG, respectively. Thus, the four triangles EFG, KEF, KFG, and KEG are equilateral. Thus, a pyramid, whose base is triangle EFG, and apex the point K, has been constructed from four equilateral triangles.

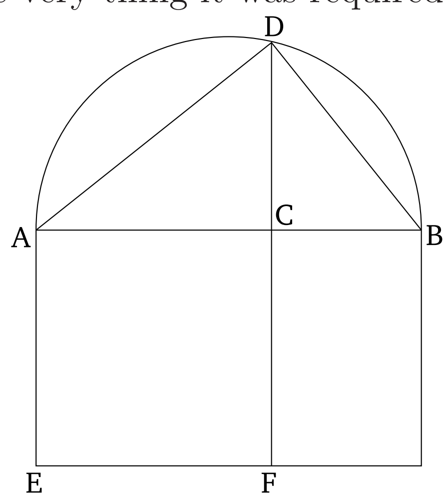
So, it is also necessary to enclose it in the given sphere, and to show that the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.

For let the straight-line HL have been produced in a straight-line with KH, and let HL be made equal to CB. And since as AC (is) to CD, so CD (is) to CB [Prop. 6.8 corr.], and AC (is) equal to KH, and CD to HE, and CB to HL, thus as KH is to HE, so EH (is) to HL. Thus, the (rectangle contained) by KH and HL is equal to the (square) on EH [Prop. 6.17]. And each of the angles KHE and EHL is a right-angle. Thus, the semi-circle drawn on KL will also pass through E [inasmuch as if we join EL then the angle LEK becomes a right-angle, on account of triangle ELK becoming equiangular to each of the triangles ELH and EHK [Props. 6.8, 3.31]]. So, if KL remains (fixed), and the semi-circle is carried around, and again established at the same (position) from which it began to be moved, it will also pass through points F and G, (because) if FL and LG are joined, the angles at F and G will similarly become right-angles. And the pyramid will have been enclosed by the given sphere. For the diameter, KL, of the sphere is equal to the diameter, AB, of the given sphere—inasmuch as KH was made equal to AC, and HL to CB.

So, I say that the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.

For since AC is double CB, AB is thus triple BC. Thus, via conversion, BA is one and a half times AC. And as BA (is) to AC, so the (square) on BA (is) to the (square) on AD [inasmuch as if DB is joined then as BA is to AD, so DA (is) to AC, on account of the similarity of triangles DAB and DAC. And as the first is to the third (of four proportional magnitudes), so the (square) on the first (is) to the (square) on the second.] Thus, the (square) on BA (is) also one and a half times the (square) on AD. And BA is the diameter of the given sphere, and AD (is) equal to the side of the pyramid.

Thus, the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.† (Which is) the very thing it was required to show.



Lemma

It must be shown that as AB is to BC, so the (square) on AD (is) to the (square) on DC.

For, let the figure of the semi-circle have been set out, and let DB have been joined. And let the square EC have been described on AC. And let the parallelogram FB have been completed. Therefore, since, on account of triangle DAB being equiangular to triangle DAC [Props. 6.8, 6.4], (proportionally) as BA is to AD, so DA (is) to AC, the (rectangle contained) by BA and

AC is thus equal to the (square) on AD [Prop. 6.17]. And since as AB is to BC, so EB (is) to BF [Prop. 6.1]. And EB is the (rectangle contained) by BA and AC—for EA (is) equal to AC. And BF the (rectangle contained) by AC and CB. Thus, as AB (is) to BC, so the (rectangle contained) by BA and AC (is) to the (rectangle contained) by AC and CB. And the (rectangle contained) by BA and AC is equal to the (square) on AD, and the (rectangle contained) by ACB (is) equal to the (square) on DC. For the perpendicular DC is the mean proportional to the pieces of the base, AC and CB, on account of ADB being a right-angle [Prop. 6.8 corr.]. Thus, as AB (is) to BC, so the (square) on AD (is) to the (square) on DC. (Which is) the very thing it was required to show.