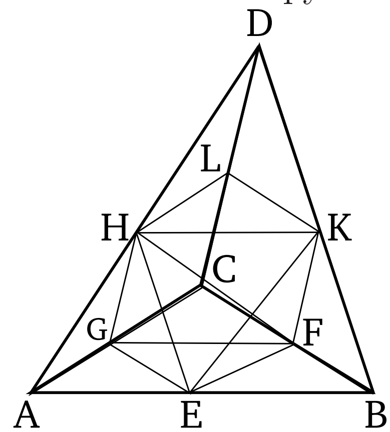
Book 12
Proposition 3

Any pyramid having a triangular base is divided into two pyramids having triangular bases (which are) equal, similar to one another, and [similar] to the whole, and into two equal prisms. And the (sum of the) two prisms is greater than half of the whole pyramid.



Let there be a pyramid whose base is triangle ABC, and (whose) apex (is) point D. I say that pyramid $ABCD$ is divided into two pyramids having triangular bases (which are) equal to one another, and similar to the whole, and into two equal prisms. And the (sum of the) two prisms is greater than half of the whole pyramid.

For let AB, BC, CA, AD, DB, and DC have been cut in half at points E, F, G, H, K, and L (respectively). And let HE, EG, GH, HK, KL, LH, KF, and FG have been joined. Since AE is equal to EB, and AH to DH, EH is thus parallel to DB [Prop. 6.2]. So, for the same (reasons), HK is also parallel to AB. Thus, $HEBK$ is a parallelogram. Thus, HK is equal to EB [Prop. 1.34]. But, EB is equal to EA. Thus, AE is also equal to HK.

And AH is also equal to HD. So the two (straight-lines) EA and AH are equal to the two (straight-lines) KH and HD, respectively. And angle EAH (is) equal to angle KHD [Prop. 1.29]. Thus, base EH is equal to base KD [Prop. 1.4]. Thus, triangle AEH is equal and similar to triangle HKD [Prop. 1.4]. So, for the same (reasons), triangle AHG is also equal and similar to triangle HLD. And since EH and HG are two straight-lines joining one another (which are respectively) parallel to two straight-lines joining one another, KD and DL, not being in the same plane, they will contain equal angles [Prop. 11.10]. Thus, angle EHG is equal to angle KDL. And since the two straight-lines EH and HG are equal to the two straight-lines KD and DL, respectively, and angle EHG is equal to angle KDL, base EG [is] thus equal to base KL [Prop. 1.4]. Thus, triangle EHG is equal and similar to triangle KDL. So, for the same (reasons), triangle AEG is also equal and similar to triangle HKL. Thus, the pyramid whose base is triangle AEG, and apex the point H, is equal and similar to the pyramid whose base is triangle HKL, and apex the point D [Def. 11.10]. And since HK has been drawn parallel to one of the sides, AB, of triangle ADB, triangle ADB is equiangular to triangle DHK [Prop. 1.29], and they have proportional sides. Thus, triangle ADB is
So, for the same (reasons), triangle DBC is also similar to triangle DKL, and ADC to DLH. And since two straight-lines joining one another, BA and AC, are parallel to two straight-lines joining one another, KH and HL, not in the same plane, they will contain equal angles [Prop. 11.10]. Thus, angle BAC is equal to (angle)

KHL. And as BA is to AC, so KH (is) to HL. Thus, triangle ABC is similar to triangle HKL [Prop. 6.6]. And, thus, the pyramid whose base is triangle ABC, and apex the point D, is similar to the pyramid whose base is triangle HKL, and apex the point D [Def. 11.9]. But, the pyramid whose base [is] triangle HKL, and apex the point D, was shown (to be) similar to the pyramid whose base is triangle AEG, and apex the point H. Thus, each of the pyramids $AEGH$ and $HKLD$ is similar to the whole pyramid $ABCD$.

And since BF is equal to FC, parallelogram $EBFG$ is double triangle GFC [Prop. 1.41]. And since, if two prisms (have) equal heights, and the former has a parallelogram as a base, and the latter a triangle, and the parallelogram (is) double the triangle, then the prisms are equal [Prop. 11.39], the prism contained by the two triangles BKF and EHG, and the three parallelograms $EBFG$, $EBKH$, and $HKFG$, is thus equal to the prism contained by the two triangles GFC and HKL, and the three parallelograms $KFCL$, $LCGH$, and $HKFG$. And (it is) clear that each of the prisms whose base (is) parallelogram $EBFG$, and opposite (side) straight-line HK, and whose base (is) triangle GFC, and opposite (plane) triangle HKL, is greater than each of the pyramids whose bases are triangles AEG and HKL, and apexes the points H and D (respectively), inasmuch as, if we [also] join the straight-lines EF and EK then the prism whose base (is) parallelogram $EBFG$, and opposite (side) straight-line HK, is greater than the pyramid whose base (is) triangle EBF, and apex the point

K. But the pyramid whose base (is) triangle EBF, and apex the point K, is equal to the pyramid whose base is triangle AEG, and apex point H. For they are contained by equal and similar planes. And, hence, the prism whose base (is) parallelogram $EBFG$, and opposite (side) straight-line HK, is greater than the pyramid whose base (is) triangle AEG, and apex the point H. And the prism whose base is parallelogram $EBFG$, and opposite (side) straight-line HK, (is) equal to the prism whose base (is) triangle GFC, and opposite (plane) triangle HKL. And the pyramid whose base (is) triangle AEG, and apex the point H, is equal to the pyramid whose base (is) triangle HKL, and apex the point D. Thus, the (sum of the) aforementioned two prisms is greater than the (sum of the) aforementioned two pyramids, whose bases (are) triangles AEG and HKL, and apexes the points H and D (respectively).

Thus, the whole pyramid, whose base (is) triangle ABC, and apex the point D, has been divided into two pyramids (which are) equal to one another [and similar to the whole], and into two equal prisms. And the (sum of the) two prisms is greater than half of the whole pyramid. (Which is) the very thing it was required to show.