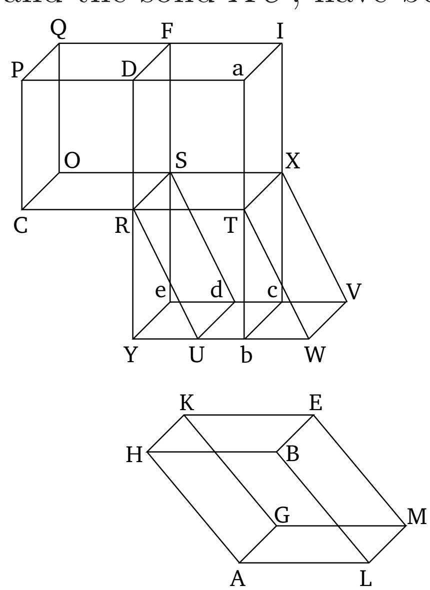
Book 11
Proposition 31

Parallelepiped solids which are on equal bases, and (have) the same height, are equal to one another.

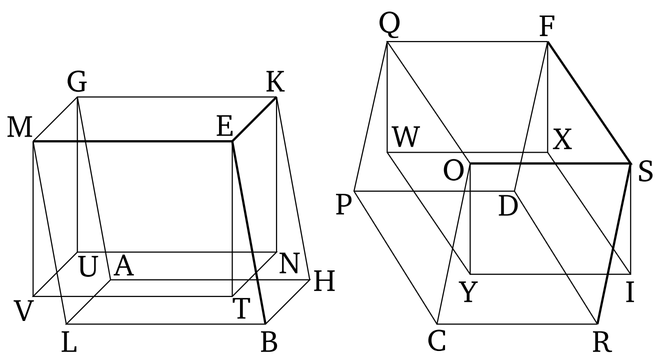
Let the parallelepiped solids AE and CF be on the equal bases AB and CD (respectively), and (have) the same height. I say that solid AE is equal to solid CF.

So, let the (straight-lines) standing up, HK, BE, AG, LM, PQ, DF, CO, and RS, first of all, be at right-angles to the bases AB and CD. And let RT have been produced in a straight-line with CR. And let (angle) TRU, equal to angle ALB, have been constructed on the straight-line RT, at the point R on it [Prop. 1.23]. And let RT be made equal to AL, and RU to LB. And let the base RW, and the solid XU, have been completed.



And since the two (straight-lines) TR and RU are equal to the two (straight-lines) AL and LB (respectively), and they contain equal angles, parallelogram RW is thus equal and similar to parallelogram HL [Prop. 6.14]. And, again, since AL is equal to RT, and LM to RS, and they contain right-angles, parallelogram RX is thus equal and similar to parallelogram AM [Prop. 6.14]. So, for the same (reasons), LE is also equal and similar to SU. Thus, three parallelograms of solid AE are equal and similar to three parallelograms of solid XU. But, the three (faces of the former solid) are equal and similar to the three opposite (faces), and the three (faces of the latter solid) to the three opposite (faces) [Prop. 11.24]. Thus, the whole parallelepiped solid AE is equal to the whole parallelepiped solid XU [Def. 11.10]. Let DR and WU have been drawn across, and let them have met one another at Y. And let aTb have been drawn through T parallel to DY. And let PD have been produced to a. And let the solids YX and RI have been completed. So, solid XY, whose base is parallelogram RX, and opposite (face) Yc, is equal to solid XU, whose base (is) parallelogram RX, and opposite (face) UV. For they are on the same base RX, and (have) the same height, and the (ends of the straight-lines) standing up in them, RY, RU, Tb, TW, Se, Sd, Xc and XV, are on the same straight-lines, YW and eV [Prop. 11.29]. But, solid XU is equal to AE. Thus, solid XY is also equal to solid AE. And since parallelogram $RUWT$ is equal to parallelogram YT. For they are on the same base RT, and between the same parallels RT and YW [Prop. 1.35]. But, $RUWT$ is equal to CD, since (it is) also (equal) to AB. Parallelogram YT is thus also equal to CD. And

DT is another (parallelogram). Thus, as base CD is to DT, so YT (is) to DT [Prop. 5.7]. And since the parallelepiped solid CI has been cut by the plane RF, which is parallel to the opposite planes (of CI), as base CD is to base DT, so solid CF (is) to solid RI [Prop. 11.25]. So, for the same (reasons), since the parallelepiped solid YI has been cut by the plane RX, which is parallel to the opposite planes (of YI), as base YT is to base TD, so solid YX (is) to solid RI [Prop. 11.25]. But, as base CD (is) to DT, so YT (is) to DT. And, thus, as solid CF (is) to solid RI, so solid YX (is) to solid RI. Thus, solids CF and YX each have the same ratio to RI [Prop. 5.11]. Thus, solid CF is equal to solid YX [Prop. 5.9]. But, YX was show (to be) equal to AE. Thus, AE is also equal to CF.



And so let the (straight-lines) standing up, AG, HK, BE, LM, CO, PQ, DF, and RS, not be at right-angles to the bases AB and CD. Again, I say that solid AE (is) equal to solid CF. For let KN, ET, GU, MV, QW, FX, OY, and SI have been drawn from points K, E, G, M, Q, F, O, and S (respectively) perpendicular to the reference plane (*i.e.*, the plane of the bases AB and CD), and let them have met the plane at points N, T,

U, V, W, X, Y, and I (respectively). And let NT, NU, UV, TV, WX, WY, YI, and IX have been joined. So solid KV is equal to solid QI. For they are on the equal bases KM and QS, and (have) the same height, and the (straight-lines) standing up in them are at right-angles to their bases (see first part of proposition). But, solid KV is equal to solid AE, and QI to CF. For they are on the same base, and (have) the same height, and the (straight-lines) standing up in them are not on the same straight-lines [Prop. 11.30]. Thus, solid AE is also equal to solid CF.

Thus, parallelepiped solids which are on equal bases, and (have) the same height, are equal to one another. (Which is) the very thing it was required to show.