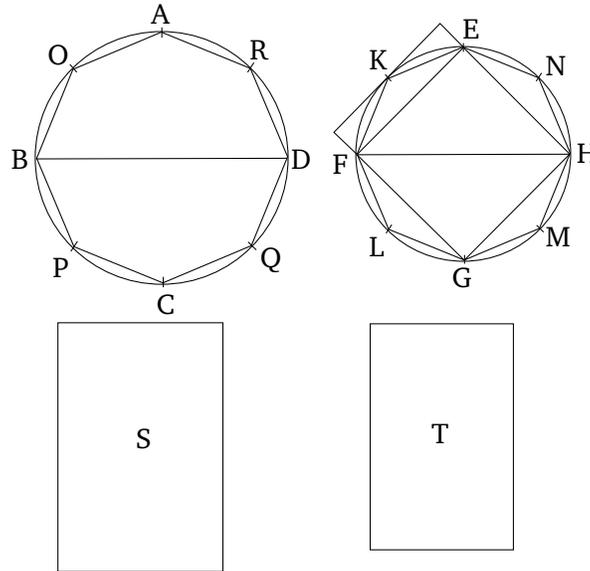


# Book 12

## Proposition 2

Circles are to one another as the squares on (their) diameters.

Let  $ABCD$  and  $EFGH$  be circles, and [let]  $BD$  and  $FH$  [be] their diameters. I say that as circle  $ABCD$  is to circle  $EFGH$ , so the square on  $BD$  (is) to the square on  $FH$ .



For if the circle  $ABCD$  is not to the (circle)  $EFGH$ , as the square on  $BD$  (is) to the (square) on  $FH$ , then as the (square) on  $BD$  (is) to the (square) on  $FH$ , so circle  $ABCD$  will be to some area either less than, or greater than, circle  $EFGH$ . Let it, first of all, be (in that ratio) to (some) lesser (area),  $S$ . And let the square  $EFGH$  have been inscribed in circle  $EFGH$  [Prop. 4.6]. So the inscribed square is greater than half of circle  $EFGH$ , inasmuch as if we draw tangents to the circle through the

points  $E$ ,  $F$ ,  $G$ , and  $H$ , then square  $EFGH$  is half of the square circumscribed about the circle [Prop. 1.47], and the circle is less than the circumscribed square. Hence, the inscribed square  $EFGH$  is greater than half of circle  $EFGH$ . Let the circumferences  $EF$ ,  $FG$ ,  $GH$ , and  $HE$  have been cut in half at points  $K$ ,  $L$ ,  $M$ , and  $N$  (respectively), and let  $EK$ ,  $KF$ ,  $FL$ ,  $LG$ ,  $GM$ ,  $MH$ ,  $HN$ , and  $NE$  have been joined. And, thus, each of the triangles  $EKF$ ,  $FLG$ ,  $GMH$ , and  $HNE$  is greater than half of the segment of the circle about it, inasmuch as if we draw tangents to the circle through points  $K$ ,  $L$ ,  $M$ , and  $N$ , and complete the parallelograms on the straight-lines  $EF$ ,  $FG$ ,  $GH$ , and  $HE$ , then each of the triangles  $EKF$ ,  $FLG$ ,  $GMH$ , and  $HNE$  will be half of the parallelogram about it, but the segment about it is less than the parallelogram. Hence, each of the triangles  $EKF$ ,  $FLG$ ,  $GMH$ , and  $HNE$  is greater than half of the segment of the circle about it. So, by cutting the circumferences remaining behind in half, and joining straight-lines, and doing this continually, we will (eventually) leave behind some segments of the circle whose (sum) will be less than the excess by which circle  $EFGH$  exceeds the area  $S$ . For we showed in the first theorem of the tenth book that if two unequal magnitudes are laid out, and if (a part) greater than a half is subtracted from the greater, and (if from) the remainder (a part) greater than a half (is subtracted), and this happens continually, then some magnitude will (eventually) be left which will be less than the lesser laid out magnitude [Prop. 10.1]. Therefore, let the (segments) have been left, and let the (sum of the) segments of the circle  $EFGH$  on  $EK$ ,  $KF$ ,

$FL$ ,  $LG$ ,  $GM$ ,  $MH$ ,  $HN$ , and  $NE$  be less than the excess by which circle  $EFGH$  exceeds area  $S$ . Thus, the remaining polygon  $EKFLGMHN$  is greater than area  $S$ . And let the polygon  $AOBPCQDR$ , similar to the polygon  $EKFLGMHN$ , have been inscribed in circle  $ABCD$ . Thus, as the square on  $BD$  is to the square on  $FH$ , so polygon  $AOBPCQDR$  (is) to polygon  $EKFLGMHN$  [Prop. 12.1]. But, also, as the square on  $BD$  (is) to the square on  $FH$ , so circle  $ABCD$  (is) to area  $S$ . And, thus, as circle  $ABCD$  (is) to area  $S$ , so polygon  $AOBPGQDR$  (is) to polygon  $EKFLGMHN$  [Prop. 5.11]. Thus, alternately, as circle  $ABCD$  (is) to the polygon (inscribed) within it, so area  $S$  (is) to polygon  $EKFLGMHN$  [Prop. 5.16]. And circle  $ABCD$  (is) greater than the polygon (inscribed) within it. Thus, area  $S$  is also greater than polygon  $EKFLGMHN$ . But, (it is) also less. The very thing is impossible. Thus, the square on  $BD$  is not to the (square) on  $FH$ , as circle  $ABCD$  (is) to some area less than circle  $EFGH$ . So, similarly, we can show that the (square) on  $FH$  (is) not to the (square) on  $BD$  as circle  $EFGH$  (is) to some area less than circle  $ABCD$  either.

So, I say that neither (is) the (square) on  $BD$  to the (square) on  $FH$ , as circle  $ABCD$  (is) to some area greater than circle  $EFGH$ .

For, if possible, let it be (in that ratio) to (some) greater (area),  $S$ . Thus, inversely, as the square on  $FH$  [is] to the (square) on  $DB$ , so area  $S$  (is) to circle  $ABCD$  [Prop. 5.7 corr.]. But, as area  $S$  (is) to circle  $ABCD$ , so circle  $EFGH$  (is) to some area less than circle  $ABCD$  (see lemma). And, thus, as the (square) on  $FH$  (is) to

the (square) on  $BD$ , so circle  $EFGH$  (is) to some area less than circle  $ABCD$  [Prop. 5.11]. The very thing was shown (to be) impossible. Thus, as the square on  $BD$  is to the (square) on  $FH$ , so circle  $ABCD$  (is) not to some area greater than circle  $EFGH$ . And it was shown that neither (is it in that ratio) to (some) lesser (area). Thus, as the square on  $BD$  is to the (square) on  $FH$ , so circle  $ABCD$  (is) to circle  $EFGH$ .

Thus, circles are to one another as the squares on (their) diameters. (Which is) the very thing it was required to show.

## Lemma

So, I say that, area  $S$  being greater than circle  $EFGH$ , as area  $S$  is to circle  $ABCD$ , so circle  $EFGH$  (is) to some area less than circle  $ABCD$ .

For let it have been contrived that as area  $S$  (is) to circle  $ABCD$ , so circle  $EFGH$  (is) to area  $T$ . I say that area  $T$  is less than circle  $ABCD$ . For since as area  $S$  is to circle  $ABCD$ , so circle  $EFGH$  (is) to area  $T$ , alternately, as area  $S$  is to circle  $EFGH$ , so circle  $ABCD$  (is) to area  $T$  [Prop. 5.16]. And area  $S$  (is) greater than circle  $EFGH$ . Thus, circle  $ABCD$  (is) also greater than area  $T$  [Prop. 5.14]. Hence, as area  $S$  is to circle  $ABCD$ , so circle  $EFGH$  (is) to some area less than circle  $ABCD$ . (Which is) the very thing it was required to show.