Hyperbolic Geometry of 2+1 Spacetime: Static Hypertext Version

In order to develop the physical interpretation of the conic construction that we made on the previous page, we will now replace the 3-dimensional Euclidean geometry of $\mathbb{R}^3$ with 2+1-dimensional Hyperbolic geometry. That geometry is determined by the "Hyperbolic metric" for $\mathbb{R}^3$, as opposed to the Euclidean metric.

Like the Euclidean metric, it is defined by a non-degenerate inner product, but unlike that metric, the inner product is not positive definite, as we shall explain below. In fact, there is a 2-dimensional stratified set of vectors called the "light cone" with the property that their "lengths" are zero. This lends the geometry a distinctive and interesting character. Still, many of the familiar properties of Euclidean geometry have their analog here. And in particular, as we shall show on this page, the ellipses that we constructed earlier by slicing the standard cone with a plane, may also be realized by forming the intersection of two light cones.

This is of course a preliminary for our physical interpretation of the focus-locus and focus-directrix properties. In the **Thought Experiment** and **Interpretation of the Experiment** sections, we will take the third step, and interpret that property using Special Relativity restricted to a 2+1-dimensional spacetime.

The geometric structure of $\mathbb{R}^3$ that we will use from now on is determined by an inner product. We continue to use the (x, y, t) Cartesian coordinates to specify that inner product, though it is understood that the inner product itself is an underlying structure that is independent of any particular choice of coordinates. That structure remains covariant under a wide class of linear transformations (Lorentz transformations) that preserve the inner product, just as the Euclidean geometry remains covariant under all rotations and inversions across planes. The inner product is defined, then, as follows:

Suppose that $W = (x_1, y_1, t_1)$ *and* $Z = (x_2, y_2, t_2)$. We will use the words "points" and "events" interchangeably in anticipation of the discussion to come later.

Then say that the inner product of W with Z, which we shall denote $\langle W, Z \rangle$ is

$$\langle W, Z \rangle = x_1x_2 + y_1y_2 - t_1t_2 \tag{4.1}$$

Note that this implies that the light cone at the origin $(0,0,0)$ is defined as the set of vectors W such that $\langle W, W \rangle = 0$. If we denote by X the vector $(1,0,0)$, by Y the vector $(0,1,0)$ and by T the vector $(0,0,1)$, then in this hyperbolic metric:

$$\begin{aligned} \langle X,X \rangle &= \langle Y,Y \rangle = 1 \text{ and } \langle T,T \rangle = -1 \\ \langle X,Y \rangle &= \langle Y,T \rangle = \langle X,T \rangle = 0 \end{aligned} \tag{4.2}$$

We use the suggestive name T for the third vector because in the physical interpretation, in which ***t*** represents the time, the name T will do double duty as the vector T which is the event of the *first clock tick on the world-line of a special (stationary) observer,* and as the name of the stationary observer itself. The vectors $X, Y,$ and T are "orthonormal" in this metric in the above sense.

This "hyperbolic" metric defines, for each pair of events: W and Z in $\mathbb{R}^3$ a number: $\langle W-Z, W-Z \rangle$. This number may be positive, negative, or zero. We'll call that number $\langle W-Z, W-Z \rangle$ the "hyperbolic interval" between events W and Z. Obviously this is equal to $\langle Z-W, Z-W \rangle$. The number is analogous to the "squared distance" between points in the Euclidean metric.

We take the liberty of coloring our prose a little (and anticipating the physics somewhat) by using terms like "light rays", "signals", "clocks" and "observers". We will say more about these ideas on the **Clocks, Light Rays and Rulers** page. If the hyperbolic interval is zero, it means that a ray of light connects W to Z. If negative, it means that W and Z are *causally connected,* in the sense that some inertial observer may go from one of these to the other: each lies in the interior of the light cone of the other. And this means that a slower-than-light signal

may pass from one to the other, so that one definitely precedes the other.

If the hyperbolic interval is positive, it means that each event lies in the exterior of the other's light cone, and the events are not causally connected. No signal may pass from one to the other, and for some inertial observers, W precedes Z, while for others, W follows Z, and for yet others, events W and Z are simultaneous.

The following physical aside is the basic physical postulate that Einstein set down for special relativity (3+1 Hyperbolic geometry), defining, in a sense, the class of allowable geometric transformations from one inertial observer to another: The hyperbolic interval separating two events is the same (number) no matter which coordinate system of an inertial observer is used to measure the coordinates of the events. This is true as long as all inertial observers choose compatible units of measure, and use those units for all measurements. This means that it must be possible to synchronize their clocks when they are pairwise stationary with respect to one another, and that each measures the speed of light to be one unit distance per unit time. When they are in uniform motion with respect to each other, each uses his own system of coordinates to describe events, but they still measure the same hyperbolic interval between any two events. In fact, it is possible for an observer to measure this interval using *clocks* and *light rays* alone. An experiment on the next page: **Clocks, Light Rays and Rulers**, will allow you to see that for yourself.

In order to discuss the plane-slicing-cone construction in a physically unified way, we introduce some geometric lemmas. These lemmas generalize some obvious facts about ordinary $\mathbb{R}^3$ with its Euclidean metric. They are rather trivial, but they point the way to the physical interpretation of this geometric operation.

Lemma 1: (Hyperbolic orthogonal bisector)

Suppose we are given two distinct events, X and Y, in 2+1 space with its hyperbolic metric: $\langle , \rangle$. Let W be the midpoint of the segment they determine. The set of vectors Z with the property that $\langle Z - W , Y - X \rangle = 0$ is a plane. This plane is the ***orthogonal bisector*** of the segment.

Proof: There is no loss of generality if we assume that W is the origin and that $Y = -X$.

Therefore, we seek the Z with the property that $\langle Z , X \rangle = 0$ with $X = (x_1, y_1, t_1)$ non-zero. Since the hyperbolic inner product (4.1) is non-degenerate, this is just the kernel of the functional $Z \rightarrow \langle Z, X \rangle$

$$(z_1, z_2, z_3) \rightarrow z_1 x_1 + z_2 y_1 - z_3 t_1$$

that is, the events (z_1, z_2, z_3) such that

$$z_1 x_1 + z_2 y_1 - z_3 t_1 = 0$$

and the result is immediate. ♦

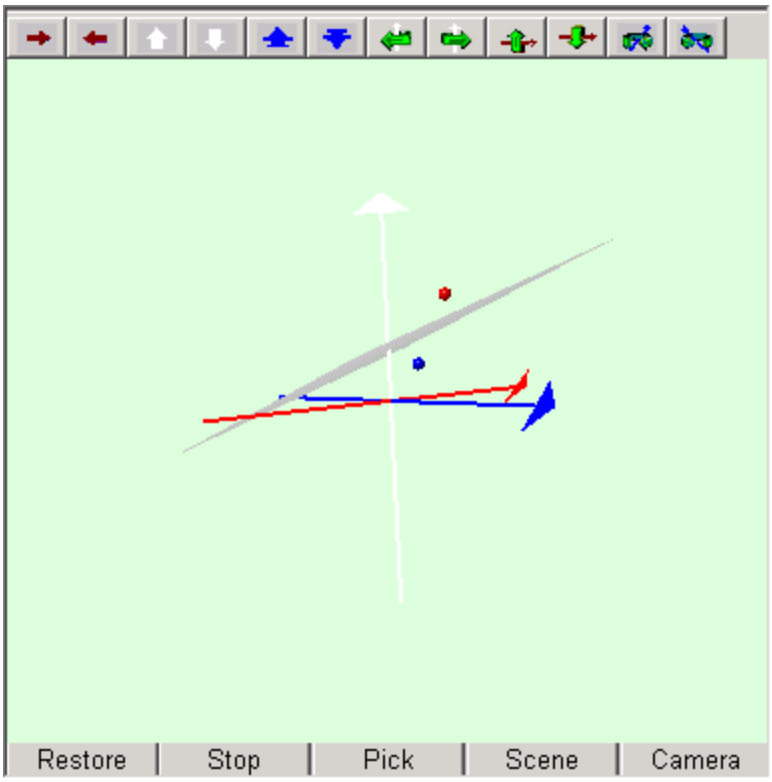
Of course, the hyperbolic orthogonal bisector of a segment does not *appear* perpendicular to the segment, as it would be in the Euclidean metric. For example, the "light ray" $(1, 0, 1)$ is orthogonal to itself! The orthogonal bisector of the "light segment" connecting

$$(-1, 0, -1) \text{ to } (1, 0, 1)$$

is the set of events

$$\{(z_1, z_2, z_3) \mid z_1 = z_3\}$$

This contains the segment. Generally, though, the picture might look something like:



Lemma 2:

Suppose we are given two distinct events, X and Y, in 2+1 space with its hyperbolic metric: $\langle , \rangle$. Let W be the midpoint of the segment they determine. The set of points Z with the property that the interval from Z to X is equal to the interval from Z to Y, that is, such that

$$\langle Z - X, Z - X \rangle = \langle Z - Y, Z - Y \rangle,$$

is a plane, and is in fact equal to the orthogonal bisector of the segment determined by X and Y.

Proof: Suppose Z satisfies the condition: $\langle Z - X, Z - X \rangle = \langle Z - Y, Z - Y \rangle$. Then it is easy to see that

$$\langle X, X \rangle - 2 \langle X, Z \rangle = \langle Y, Y \rangle - 2 \langle Y, Z \rangle$$

But this implies that

$$\langle X, X \rangle - \langle Y, Y \rangle = 2 \langle X - Y, Z \rangle$$

Thus

$$\langle X + Y, X - Y \rangle = \langle 2Z, X - Y \rangle$$

So it follows that

$$\langle 2Z - X - Y, X - Y \rangle = 0$$

or that (for W the midpoint of the segment)

$$\langle Z - W , X - Y \rangle = 0$$

But this puts Z on the orthogonal bisector of the segment. Each step of the argument is reversible, and so the orthogonal bisector is equal to the set of points with equal interval from X and Y. ♦

Lemma 3: (Conic Intersections)

Suppose we are given two distinct events, X and Y, in 2+1 space with its hyperbolic metric: $\langle , \rangle$. Let W be the midpoint of the segment they determine, and let plane p be the orthogonal bisector of the segment $\overline{XY}$ passing through W. Let c_X be the light cone with vertex at X, and let c_Y be the light cone with vertex at Y. Let d be the hyperbolic interval from X to Y:

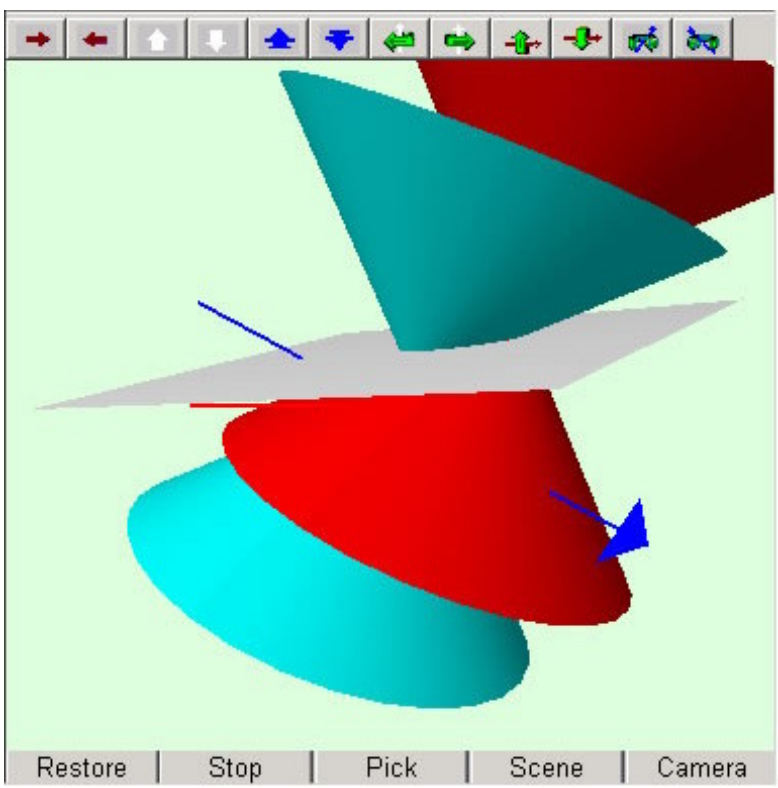
$$d = \langle X - Y , X - Y \rangle$$

Then if $d \neq 0$, then

$$c_X \cap c_Y \subset p, \text{ and}$$
$$p \cap c_X = p \cap c_Y$$

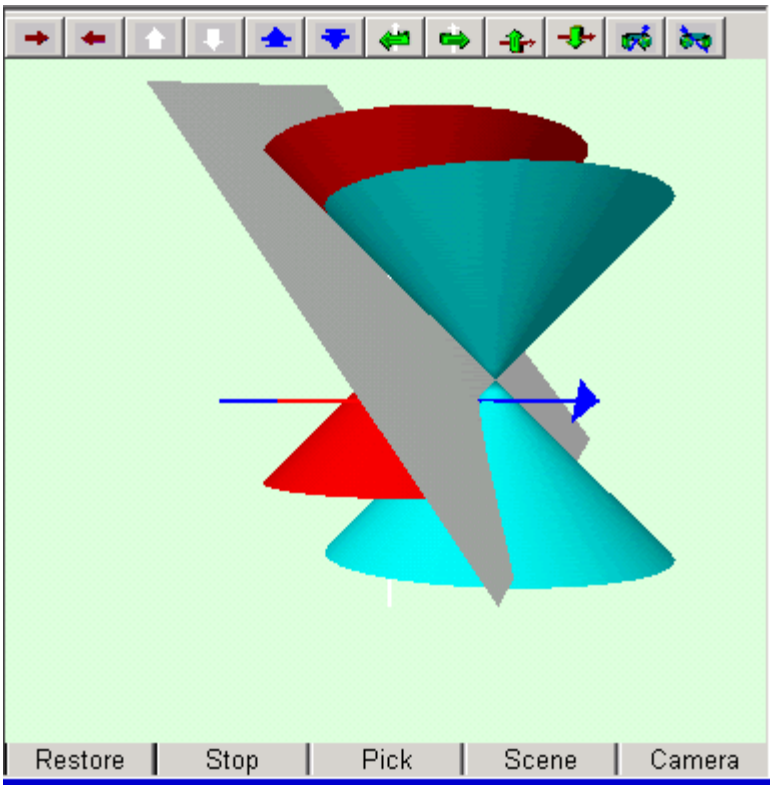
and this intersection is either an ellipse or an hyperbola. There are two cases:

a. If $d < 0$, the hyperbolic interval is "time-like". In this case, the common intersection is an ellipse.



An Elliptic Conic Intersection

b. If $d > 0$, the hyperbolic interval is "space-like". In this case, the common intersection is a hyperbola.



A Hyperbolic Conic Intersection

Proof:

We have already shown that $c_X \cap c_Y \subset p$, since events in $c_X \cap c_Y$ have zero hyperbolic interval with both X and Y and p is the orthogonal bisector plane. It will be enough to show that $p \cap c_X \subset p \cap c_Y$, and to argue from symmetry to prove the second statement.

There is no loss of generality if we assume that that W is the origin and that $Y = -X$. Since $\langle X, X \rangle \neq 0$, we may construct the linear isomorphism (analogous to the Euclidean inversion across a plane)

$$f_X : Z \rightarrow Z - 2\frac{\langle Z, X \rangle}{\langle X, X \rangle} X$$

Clearly, f_X leaves points in p (events Z such that $\langle Z, X \rangle = 0$) fixed. Also, $f_X(X) = -X$, $f_X(-X) = X$, and $f_X \circ f_X = identity$. Actually, f_X is simply the "hyperbolic inversion" across the plane orthogonal to X.

A straightforward calculation shows that

$$\langle f_X(Z), f_X(Z) \rangle = \langle Z, Z \rangle$$

so f_X preserves the hyperbolic interval. Thus, it is clear that f_X maps the light cone at the origin c_O (events Z such that $\langle Z, Z \rangle = 0$) into itself.

Further, it maps the light cone at X: c_X to the light cone at $Y = -X$: c_Y, because if $\langle Z - X, Z - X \rangle = 0$, then

$$\langle f_X(Z) + X, f_X(Z) + X \rangle = 0$$

since

$$f_X(Z - X) = f_X(Z) - f_X(X) = f_X(Z) + X$$

so

$$\langle f_X(Z) - Y, f_X(Z) - Y \rangle = 0$$

This proves that $p \cap c_X \subset p \cap c_Y$ for $Y = -X$ since f_X leaves p invariant, and maps $c_X \rightarrow c_Y$.

That is, if $u \in p \cap c_X$ then $f_X(u) = u$, but

$$u = f_X(u) \in f_X(p \cap c_X) \subset f_X(p) \cap f_X(c_X) = p \cap c_Y$$

The second part of the proof, the cases depending on the sign of d, is the following straightforward verification.

For example, suppose that $d = \langle X - Y, X - Y \rangle < 0$. Assuming again that W is the origin and that $Y = -X$, this implies that $\langle X, X \rangle < 0$. Then we want to verify that the intersection of the plane

$$p = \{ Z \mid \langle X, Z \rangle = 0 \}$$

with the light cone at X

$$c_X = \{ Z \mid \langle X - Z, X - Z \rangle = 0 \}$$

is an ellipse.

Suppose that X has coordinates (a, b, c). And suppose we represent the coordinates of Z as (x, y, t). Then we are assuming that $a^2 + b^2 < c^2$. And if Z is in the light cone at X then the argument is similar to the one we made in **Planes Intersecting Cones** for the tangent plane to the hyperboloid:

$$(x - a)^2 + (y - b)^2 = (t - c)^2$$

and if Z is in the plane

$$ax + by = ct$$

Therefore

$$x^2 + y^2 - t^2 = c^2 - a^2 - b^2 > 0$$

and this becomes on substitution,

$$x^2 + y^2 - \left(\frac{ax + by}{c} \right)^2 = c^2 - a^2 - b^2 > 0$$

or

$$\left(1-\frac{a^2}{c^2}\right)x^2+\left(1-\frac{b^2}{c^2}\right)y^2-\frac{2ab}{c^2}xy=c^2-a^2-b^2>0$$

The coefficients of x^2 and y^2 are both positive and the discriminant

$$\left(\frac{2ab}{c^2}\right)^2-4\left(1-\frac{a^2}{c^2}\right)\left(1-\frac{b^2}{c^2}\right)=-4\left(1-\frac{a^2+b^2}{c^2}\right)$$

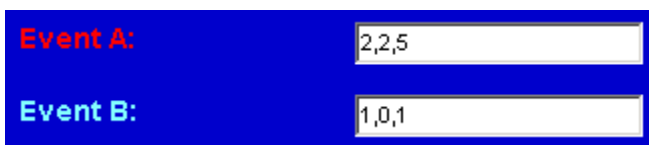
is clearly negative since $a^2+b^2<c^2$.

The argument for the hyperbolic case is similar. ♦

Description of the Exploration in the Dynamic Version

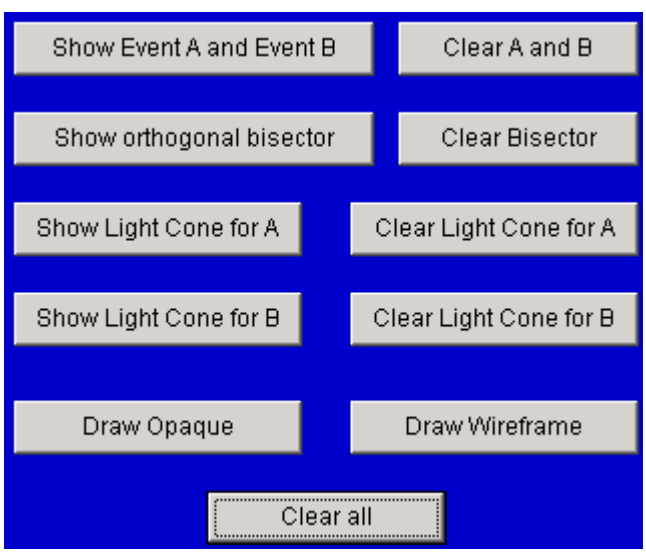
The exploration on this page invites you to experiment with the fact that conic sections are "conic intersections."

You set things up in the control panel

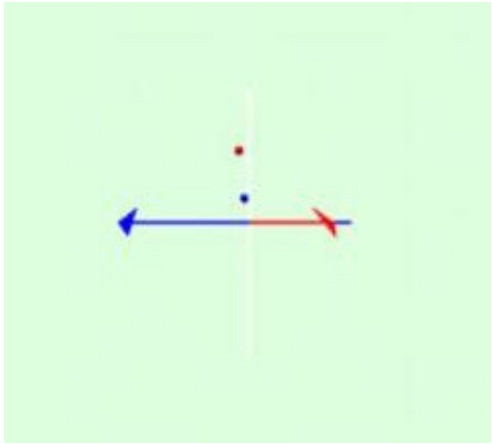


by giving the (x,y,t) coordinates of an event A and the (x,y,t) coordinates of another event B. Simply type comma-separated x,y,t without parentheses in these fields.

You will find a number of choices in the controls below.



When you press the Show Event A and Observer B button, you see the location of event A (red point) and event B (blue point) in the 2+1 screen. You may have to rotate the screen to get a perspective view. The red and blue arrows point in the ***x*** and ***y*** directions, and the white arrow points in the ***t*** (time) direction.

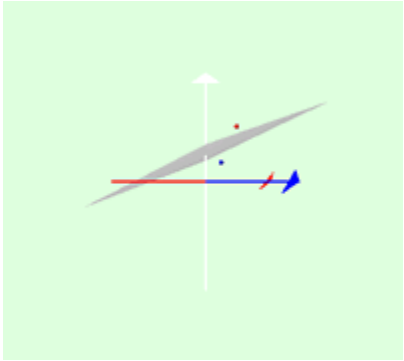


The system reports the hyperbolic interval between these events and states what sort of interval it is:

The hyperbolic interval between A and B is -8.6794919

The interval is time-like

Next, you may view the orthogonal bisector of the interval connecting A to B by pressing the Show orthogonal bisector button.



You see that the orthogonal bisector is not orthogonal in the Euclidean sense, but it is in the "hyperbolic" sense. Of course, it passes through the midpoint of the segment.

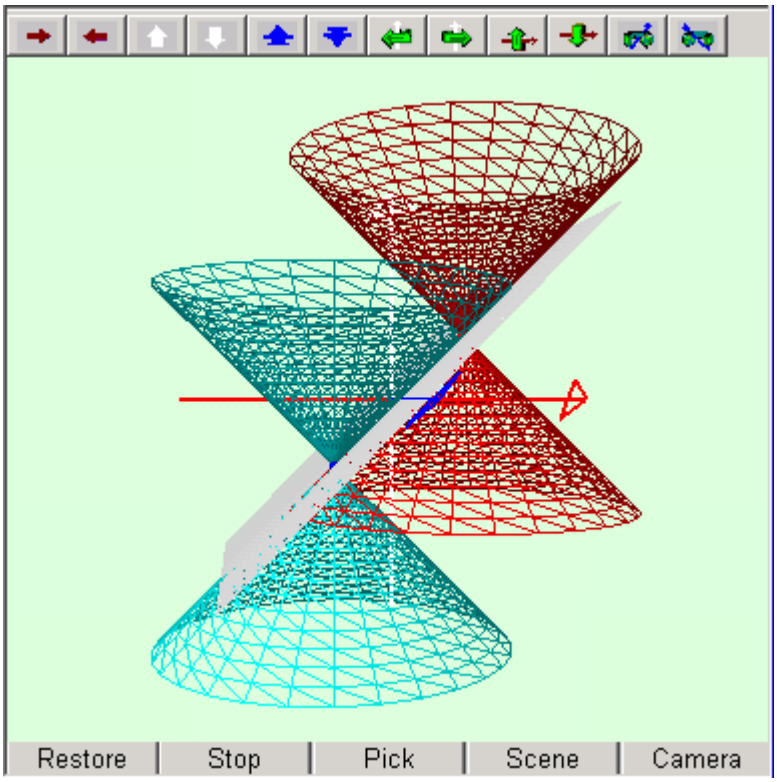
You may clear these objects by pressing the adjacent clear buttons. And you may restore the original viewpoint (after rotating) by pressing the **Restore Button** on the screen.

Now, we come to the main experiment for this page: **Conic Intersections**.

To do the experiment, press the Show Light Cone for A button. You will see the (red) light cone for event A. Next, press the Show Light Cone for B and you will see the (blue) light cone for event *B*. You have a number of choices, and you will see pictures like the ones above.

The cones will intersect in a conic section, depending on whether the hyperbolic interval was time-like or space-like. You may get a clearer view of how they intersect if you press Show orthogonal bisector because as we showed in Lemma 3 above, that plane will also contain the conic intersection.

Here, we show you an intriguing possibility, which was not discussed in Lemma 3. We choose two points with a "light-like" separation. Can you guess what the conic intersection is? It contains the points themselves!



We pressed the Draw Wireframe button to get a better view.

We discuss the idiom of Special Relativity in the next section. And in **A Thought Experiment**, we will interpret the focus-locus definition of conic sections in terms of light cones.