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Differentiate Early, Differentiate Often!

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In first-year calculus, constrained-optimization and related-rates word problems are two of the biggest stumbling blocks. In this note, I contrast the methods suggested in calculus textbooks for the solution of these two types of problems, and conclude that a different approach to constrained-optimization problems, similar to that widely used for related-rates problems, would be advantageous.

Let us first consider related-rates problems. Traditional textbooks (see, for instance, Adams [**1**, p. 235]; Edwards and Penney [**3**, p. 193]; Finney, Weir, and Giordano [**5**, p. 209]; Johnston and Mathews [**6**, p. 316]; Stewart [**8**, p. 258], and Strauss *et al.* [**9**, p. 158]) introduce these shortly after implicit differentiation. These texts all suggest that implicit differentiation of the equation relating the rates should be an early step in the solution of such a problem. Nonetheless, many students, faced with a related-rates problem, persistently avoid implicit differentiation by eliminating a variable. For instance:

Problem 1. *A ladder of length* 5 *m is sliding with one end on the ground and the other on a vertical wall. The end on the ground is sliding away from the wall at a constant rate of* 1 *m/sec. How fast is the end on the wall moving when it is* 4 *m off the ground?*

Solution A (standard). By the Pythagorean theorem, the distance x from the foot of the wall to the ladder and the height y of the top of the ladder are linked by the relation

$$x^2 + y^2 = 25; \tag{1}$$

differentiating implicitly with respect to t yields

$$x\,dx/dt + y\,dy/dt = 0. \tag{2}$$

We can now substitute the instantaneous value $y = 4$ into (1) to obtain $x = 3$; substituting these values and $dx/dt = 1$ into (2) we obtain $3 + 4dy/dt = 0$, so that $dy/dt = -3/4$ m/sec.

Solution B (avoiding implicit differentiation). Solving (1) for y, we obtain

$$y = \sqrt{25 - x^2}. \tag{3}$$

Differentiating with respect to x gives

$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{25 - x^2}}$$

and by the Chain Rule we have

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{-x}{\sqrt{25 - x^2}}\frac{dx}{dt}.$$

Substituting the instantaneous value $x = 3$ and $dx/dt = 1$, we obtain the answer.

The second approach is more difficult. Moreover, it admits two sources of error that are avoided by the first. The most common mistake on such a problem involves "freezing" one or more instances of a variable by substituting an instantaneous value before differentiation; the earlier the student differentiates, the less likely this is to happen. The other standard mistake, of course, involves incorrect differentiation of the comparatively complicated expression in (3).

All of this suggests that the traditional approach to related-rates problems is valid, and that students should be strongly encouraged to follow it. Like many other instructors, I usually take the view that students who prefer to use a certain technique, and get the right answer, should be permitted to do so. However, in this case, I feel that students who insist on avoiding implicit differentiation are not making an informed decision, even though they will probably be able to grind out the solutions to many problems.

A few weeks after related rates (depending on the textbook and course plan), students will usually encounter constrained-optimization problems. These resemble related-rates problems, not only in being presented as "word problems", but also in involving two variables on an equal footing. The usual approach in most textbooks (see, for instance, [**1**, p. 264], [**4**, p. 292], [**5**, p. 288], [**8**, pp. 331–2], and [**9**, p. 238])—and that favored by many instructors—is to use the constraint equation to eliminate one variable from the objective function, differentiate the resulting one-variable function, and find the extremum.

Problem 2. *Find the dimensions of the largest rectangle that can be inscribed in a semicircle of radius R.*

Solution A (traditional). From the constraint $x^2 + y^2 = R^2$ we get

$$y = \sqrt{R^2 - x^2}.$$

Substituting this into the area $A = 2xy$ of the rectangle, we obtain

$$A = 2x\sqrt{R^2 - x^2},$$

and differentiating this yields

$$\frac{dA}{dx} = 2\sqrt{R^2 - x^2} - \frac{2x^2}{\sqrt{R^2 - x^2}},$$

which simplifies to

$$\frac{dA}{dx} = \frac{2(R^2 - x^2) - 2x^2}{\sqrt{R^2 - x^2}}.$$

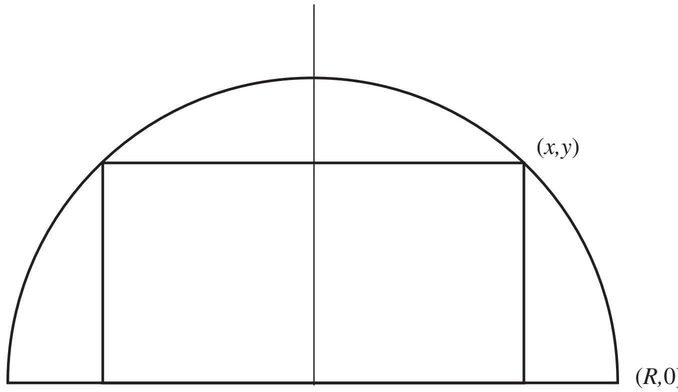

Figure 1. Maximize the area of the rectangle

Setting the numerator equal to 0 and solving, we get $2x^2 = R^2$ or $x = R/\sqrt{2}$; the edges of the rectangle are thus $R/\sqrt{2}$ and $\sqrt{2}R$.

This problem, like some others, can be simplified somewhat by maximizing the square of the area rather than the area itself. This trick, mentioned in some books, has limited application, though it is certainly worth knowing. A much more generally applicable technique is found in a few books. Implicit differentiation is used at the outset, on both the constraint and objective functions; the derivative is then eliminated to obtain the solution.

Solution B (early differentiation). Implicitly differentiating both the constraint function

$$x^2 + y^2 = R^2 \tag{4}$$

and the objective function $A = 2xy$, and setting the latter equal to 0, we get:

$$2x + 2yy' = 0$$
$$A' = 2y + 2xy' = 0.$$

Combining these, we get $2(x^2 - y^2) = 0$, whence the solution

$$x = \pm y \tag{5}$$

readily follows. Substituting this into (4) gives us $x = R/\sqrt{2}$ as before.

This is faster for two reasons. It is usually easier to differentiate a relation than to differentiate the function obtained by solving it for one variable; and the resulting equation is always linear in the derivative, so the step of eliminating the derivative is straightforward. Moreover, differentiation lowers the degree of a polynomial function, often simplifying the algebra. It is still possible that the resulting system of equations in x and y cannot be solved, but the odds are improved.

The functions in textbook constrained-optimization problems rarely go much beyond quadratics. As the complexity of the functions rises, so do the difficulties of eliminating a variable, or finding the zeros of the derivative finally obtained. Fortunately, quadratic relations are common in real-world applications, so the techniques learned do have some value.

With the early-differentiation method, it is possible to go somewhat further. The reader may like to try and compare the two methods on the problem of determining the point at which the curve $2x^3 + 3xy^2 + y^3 = 6$ comes closest to the origin. The above-mentioned trick of minimizing $x^2 + y^2$ rather than the distance itself will help here.

Early differentiation is given as an alternative method for one problem in Adams [**1**, pp. 266–7], and one in Stewart [**8**, p. 334]; but neither author suggests it as a method of first choice. Most textbooks examined do not mention it at all. Interestingly, *Schaum's Outline of Calculus*, while characteristically sparing of explanation, gives three examples of this technique [**2**, pp. 50–53] among ten worked constrained optimization problems (cf. [**7**, pp. 237–242].)

It is also worth noting that (5) gives not only a solution to the entire family of equations with different values of R, but also the general solution to the dual family of problems in which a rectangle of specified area must be inscribed in a semicircle with the smallest possible radius (this is mentioned in Adams, [**1**]). These are general features of this approach whenever the objective and constraint are both specified as values of functions; this duality will be familiar to the student who has studied linear programming, but is not commonly mentioned in first-year calculus. With early differentiation, little extra effort is needed to do so.

Implicit differentiation is, of course, an important technique in its own right, and is used heavily in subjects such as thermodynamics, mechanics, and economics. It is usually only "in the spotlight" for a comparatively short period during the first year calculus course, and students may consider it as an unimportant diversion from the main thrust of the course. Stressing it as a technique for both related-rates and constrained-optimization problems should emphasize its true importance.

Finally, the student who continues into multivariate calculus will learn to solve more advanced optimization problems using the method of Lagrange multipliers. Here, too, an important part of the technique is to do the differentiation first, rather than eliminating variables; the student who is already confident with operating in this order should find Lagrange multipliers less intimidating.

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