

- [ 3 ] Geometry Problem 34; proposed as Problem 33 in 1 (1894) 317; solution as Problem 34 in 2 (1895) 48-49.  
Both problems—a horse can graze inside the fence and then outside the fence. The solution published for outside the fence seems obviously incorrect.
- [ 4 ] Calculus Problem 37; proposed in 2 (1895) 52; solution in 2 (1895) 277-78.  
Two mules graze together, but on opposite sides of the fence.
- [ 5 ] Calculus Problem 55; proposed in 3 (1896) 148; solutions in 4 (1897) 17-18.  
Both problems again—a horse can graze outside the fence and then inside the fence. Involution enters into the solutions.
- [ 6 ] Calculus Problem 69; proposed in 5 (1898) 29; solution in 5 (1898) 111; note in 5 (1898) 177.  
Horse is tethered to a sliding ring outside an elliptical field, perhaps the most exotic problem. The note points out that the published solution is incorrect, but a correct solution is not offered.
- [ 7 ] Arithmetic Problem 93; proposed as Problem 91 in 5 (1898) 60; solution as Problem 93 in 5 (1898) 170-71.  
Another horse is tethered to the corner of a barn. But six different answers were received.
- [ 8 ] Geometry Problem 103; proposed in 5 (1898) 217; solutions in 5 (1898) 295-96.  
A horse is grazing on the edge of a circular pond. Three solutions reflect different assumptions about whether the rope can stretch across the pond. Curiously, the answer is almost the same in each case.
- [ 9 ] Calculus Problem 103; proposed in 6 (1899) 289; remark in 7 (1900) 267-68.  
Essentially the same as Calculus Problem 69—a horse is grazing outside an elliptical field tethered to a sliding ring. No solution was received and, so far as I know, these two problems remain unsolved.

Grazing goat problems and some variants from recent sources.

- [10] V. W. B., An iterative process: the goat's share revisited, *Math. Gaz.*, 65 (1981) 137-39.  
The classical goat; discussion of iterative calculator solution of the formidable-looking arcsine equation.
- [11] Howard P. Dinesman, *Superior Mathematical Puzzles*, Simon and Schuster, New York, 1968; Puzzle 8 on p. 20 and Puzzle 53 on p. 71.  
Puzzle 8 is very easy; in Puzzle 53 a goat is tethered to a circular silo.
- [12] Jordi Dou, Solution to Problem S19 (proposed by Harley Flanders), *Amer. Math. Monthly*, 88 (1981) 147.
- [13] Henry Ernest Dudeney, *Amusements in Mathematics*, Dover, New York, 1958 (reprint of 1917 edition), Problem 196 on p. 53.  
Goat grazing in an equilateral triangle; easy.
- [14] L. A. Graham, *Ingenious Mathematical Problems*, Dover, New York, 1959, p. 6.  
This has a literary twist, and is called "Mrs. Miniver's Problem."
- [15] S. I. Jones, *Mathematical Nuts*, S. I. Jones Co., 1932.  
Note the quick calculation of the area of the involute.
- [16] P. M. H. Kendall and G. M. Thomas, *Mathematical Puzzles for the Connoisseur*, Thomas Y. Crowell, New York, 1962, pp. 24-25.  
A macabre goat is tethered to a mausoleum in a circular field. A nice involute problem.
- [17] L. H. Longley-Cook, *Work This One Out*, Fawcett, 1960, Problem 69.
- [18] James E. Schultz and Bert K. Waits, A new look at some old problems in light of the hand calculator, *TYCMJ*, 10 (1979) 20-27.  
Discusses calculator solutions of three classical problems—one of which is the grazing goat. A simple derivation using sectors of a complicated-looking equation.

## Triangle Constructions with Three Located Points

**WILLIAM WERNICK**

3986 Hillman Avenue  
Bronx, NY 10463

Given a triangle  $ABC$  we can construct its medians,  $AM_a$ ,  $BM_b$ , and  $CM_c$  which are concurrent at the centroid  $G$  (FIGURE 1). Now suppose we erase almost all of this figure, leaving only the points  $A$ ,  $B$ , and  $M_a$  *in position*. Can we reconstruct the original figure? Yes, very easily, since if we double the segment  $BM_a$  we get the point  $C$ .

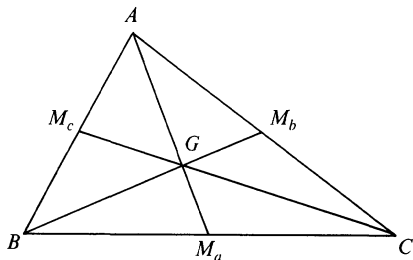


FIGURE 1

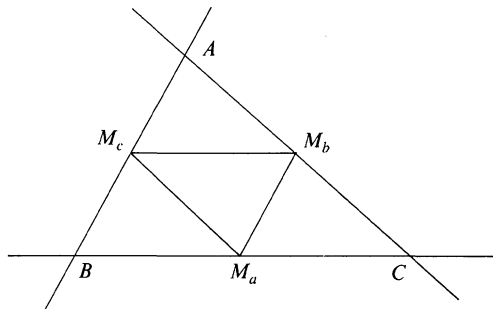


FIGURE 2

Suppose we erase all but the three midpoints  $M_a, M_b, M_c$ , in location; can we again reconstruct the original figure? One solution is to start by drawing a line through  $M_c$  parallel to the line  $M_aM_b$ , then another line through  $M_a$  parallel to the line  $M_bM_c$ . These two constructed lines will intersect in point  $B$ , a vertex of the original triangle, and the rest of the construction follows easily (FIGURE 2). We indicate this solution as follows:

$$// (M_c, M_aM_b) \cap // (M_a, M_bM_c) = B; \quad 2(BM_c) = A, \quad 2(BM_a) = C.$$

It is interesting to investigate the construction of a triangle  $ABC$ , given certain triples of located points selected from the following list of sixteen points (see FIGURES 1, 3, 4, 5):

$A, B, C, O$	Three vertices and circumcenter
$M_a, M_b, M_c, G$	Three feet of the medians, and centroid
$H_a, H_b, H_c, H$	Three feet of the altitudes, and orthocenter
$T_a, T_b, T_c, I$	Three feet of the internal angle bisectors, and incenter.

In these problems we may take two approaches: (1) we assume that a triangle has been erased, except for three located points, and we try to recover that original triangle; or (2) we choose any three distinct points of the plane and designate these as three particular points among the list of sixteen, then try to construct a triangle to fit. It is clear that the second approach includes the first and is a little more general, raising questions of constructibility and redundancy. Since it is more interesting, it is the approach we shall use.

The list in TABLE 1 is a careful compilation of exactly 139 such problems, all significantly distinct. For example, the selection of the triple of points to be two vertices and the centroid of a triangle could be listed as  $A, B, G$ ; or  $B, C, G$ ; or  $A, C, G$ , which are surely not significantly distinct. Only the first choice is listed (problem 4). Some selections of triples are redundant, that is, one of the points can be determined from the other two. For example, the triple  $A, B, M_c$  is redundant since  $M_c$  is the midpoint of segment  $AB$  (clearly any two of these three points determine the third). Such redundant triples in TABLE 1 are noted with the letter **R**; I have found three such triples in this list.

Note also that some selections, while not redundant, are still restricted as to choice of the points; the triple  $A, B, O$  is such a selection, since the circumcenter  $O$  must lie on the perpendicular

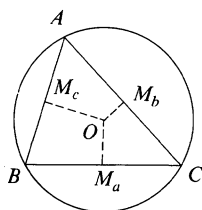


FIGURE 3

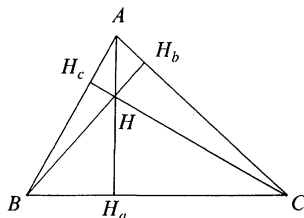


FIGURE 4

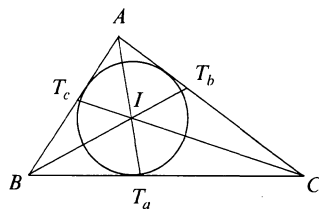


FIGURE 5

lar bisector of the side  $AB$ . If the point designated  $O$  does lie on the perpendicular bisector of the segment  $AB$ , then the third vertex  $C$  of the triangle may lie anywhere on the circumcircle with center  $O$  and radius  $OA$ . If the point designated as  $O$  does not lie on the perpendicular bisector of segment  $AB$ , then no solution is possible. In such a situation, the locus restriction gives us either infinitely many or no solutions to the problem. Twenty-two of the problems in TABLE 1 have been identified as being of this type; they are designated with the symbol L.

Of the remaining 114 problems, I have found solutions for only 65 of these, which are indicated on the list with the symbol S. Many of these 65 are quite easy and straightforward to solve. Some are harder and would challenge any student of geometry. Problems 69, 76, and 101 in particular are real challenges; they appear as Proposal 1149 in the Problems section of this issue of this *Magazine*, p. 236.

Fortunately, some problems are related, in the sense that if you solve one of them, you have ready solutions to some others. Consider, for example, problems 26, 42, and 95, whose triples of given points are as follows:

- 26:     $A, M_a, T_b$
- 42:     $A, G, T_b$
- 95:     $M_a, G, T_b$ .

Since the triple  $A, M_a, G$  is redundant (problem 21), it follows that a solution to any one of the problems 26, 42, 95 leads to immediate solutions of the other two problems. Other sets of such related problems are 27, 43, 96; 72, 79, 120; and 73, 80, 121.

Our list in TABLE 1 contains 41 seemingly independent, presently unsolved problems. It may be that some of these lead to provably impossible constructions, in which case such a proof would

1. $A, B, O$	L	29. $A, M_b, G$	S	57. $A, H, I$		85. $M_a, M_b, H_a$	S	113. $M_a, T_b, T_c$	
2. $A, B, M_a$	S	30. $A, M_b, H_a$	L	58. $A, T_a, T_b$		86. $M_a, M_b, H_c$	S	114. $M_a, T_b, I$	
3. $A, B, M_c$	R	31. $A, M_b, H_b$	L	59. $A, T_a, I$	L	87. $M_a, M_b, H$		115. $G, H_a, H_b$	
4. $A, B, G$	S	32. $A, M_b, H_c$	L	60. $A, T_b, T_c$	S	88. $M_a, M_b, T_a$		116. $G, H_a, H$	S
5. $A, B, H_a$	L	33. $A, M_b, H$	S	61. $A, T_b, I$	S	89. $M_a, M_b, T_c$		117. $G, H_a, T_a$	S
6. $A, B, H_c$	L	34. $A, M_b, T_a$	S	62. $O, M_a, M_b$	S	90. $M_a, M_b, I$		118. $G, H_a, T_b$	
7. $A, B, H$	S	35. $A, M_b, T_b$	L	63. $O, M_a, G$	S	91. $M_a, G, H_a$	L	119. $G, H_a, I$	
8. $A, B, T_a$	S	36. $A, M_b, T_c$	S	64. $O, M_a, H_a$	L	92. $M_a, G, H_b$	S	120. $G, H, T_a$	
9. $A, B, T_c$	L	37. $A, M_b, I$	S	65. $O, M_a, H_b$	S	93. $M_a, G, H$	S	121. $G, H, I$	
10. $A, B, I$	S	38. $A, G, H_a$	L	66. $O, M_a, H$	S	94. $M_a, G, T_a$	S	122. $G, T_a, T_b$	
11. $A, O, M_a$	S	39. $A, G, H_b$	S	67. $O, M_a, T_a$	L	95. $M_a, G, T_b$		123. $G, T_a, I$	
12. $A, O, M_b$	L	40. $A, G, H$	S	68. $O, M_a, T_b$		96. $M_a, G, I$		124. $H_a, H_b, H_c$	S
13. $A, O, G$	S	41. $A, G, T_a$	S	69. $O, M_a, I$	S	97. $M_a, H_a, H_b$	S	125. $H_a, H_b, H$	S
14. $A, O, H_a$	S	42. $A, G, T_b$		70. $O, G, H_a$	S	98. $M_a, H_a, H$	L	126. $H_a, H_b, T_a$	S
15. $A, O, H_b$	S	43. $A, G, I$		71. $O, G, H$	R	99. $M_a, H_a, T_a$	L	127. $H_a, H_b, T_c$	
16. $A, O, H$	S	44. $A, H_a, H_b$	S	72. $O, G, T_a$		100. $M_a, H_a, T_b$		128. $H_a, H_b, I$	
17. $A, O, T_a$	S	45. $A, H_a, H$	L	73. $O, G, I$		101. $M_a, H_a, I$	S	129. $H_a, H, T_a$	L
18. $A, O, T_b$	S	46. $A, H_a, T_a$	L	74. $O, H_a, H_b$		102. $M_a, H_b, H_c$	S	130. $H_a, H, T_b$	
19. $A, O, I$	S	47. $A, H_a, T_b$	S	75. $O, H_a, H$	S	103. $M_a, H_b, H$	S	131. $H_a, H, I$	
20. $A, M_a, M_b$	S	48. $A, H_a, I$	S	76. $O, H_a, T_a$	S	104. $M_a, H_b, T_a$	S	132. $H_a, T_a, T_b$	
21. $A, M_a, G$	R	49. $A, H_b, H_c$	S	77. $O, H_a, T_b$		105. $M_a, H_b, T_b$	S	133. $H_a, T_a, I$	S
22. $A, M_a, H_a$	L	50. $A, H_b, H$	L	78. $O, H_a, I$		106. $M_a, H_b, T_c$		134. $H_a, T_b, T_c$	
23. $A, M_a, H_b$	S	51. $A, H_b, T_a$	S	79. $O, H, T_a$		107. $M_a, H_b, I$		135. $H_a, T_b, I$	
24. $A, M_a, H$	S	52. $A, H_b, T_b$	L	80. $O, H, I$		108. $M_a, H, T_a$		136. $H, T_a, T_b$	
25. $A, M_a, T_a$	S	53. $A, H_b, T_c$	S	81. $O, T_a, T_b$		109. $M_a, H, T_b$		137. $H, T_a, I$	
26. $A, M_a, T_b$		54. $A, H_b, I$	S	82. $O, T_a, I$		110. $M_a, H, I$		138. $T_a, T_b, T_c$	
27. $A, M_a, I$		55. $A, H, T_a$	S	83. $M_a, M_b, M_c$	S	111. $M_a, T_a, T_b$		139. $T_a, T_b, I$	S
28. $A, M_b, M_c$	S	56. $A, H, T_b$		84. $M_a, M_b, G$	S	112. $M_a, T_a, I$	S		

TABLE 1. For each of the 139 triples of points listed, construct the corresponding triangle  $ABC$ . Problems solved by the author are noted with an S, L, or R to designate a solution triangle  $ABC$ , a locus-dependent solution, or a redundant triple, respectively.

constitute a solution. However, I have the firm conviction that they can all be “done,” and that eventually every one of the 139 problems will have the appropriate **R** or **L** or **S** designation. I welcome any companionship and friendly competition on the last 41 (which incidentally is a prime number with pleasant associations).

I close with a few solutions to whet your appetite.

**PROBLEM 18.** Given points  $A, O, T_b$ .

*Solution.*  $\perp(O, AT_b) \cap \text{Cir}(O, OA) = K$ ;  $KT_b \cap \text{Cir}(O, OA) = B$ ;  $AT_b \cap \text{Cir}(O, OA) = C$ . In words, the perpendicular from  $O$  to the line  $AT_b$  will meet the circle with center  $O$  and radius  $OA$  at the point  $K$ ; the line  $KT_b$  will meet the circle with center  $O$  and radius  $OA$  at the point  $B$ ; (now it's your turn)  $AT_b \cap \text{Cir}(O, OA) = C$ .

**PROBLEM 49.** Given points  $A, H_b, H_c$ .

*Solution.*  $\text{Cir}(A, H_b, H_c) = \text{Cir}(K)$ ;  $\text{Diam Cir}(K) = AH$ ;  $H_bH \cap AH_c = B$ ;  $H_cH \cap AH_b = C$ . That is, the circle which goes through points  $A, H_b$ , and  $H_c$  (the circumcircle of triangle  $AH_bH_c$ ) we call circle  $K$ ; the diameter of circle  $K$  which has one end at  $A$  has the other end at the point  $H$ ; the line  $H_bH$  will meet the line  $AH_c$  at the point  $B$ ; and now  $H_cH \cap AH_b = C$ .

**PROBLEM 65.** Given points  $O, M_a, H_b$ .

*Solution.*  $\perp(M_a, M_aO) = xM_a y$ ;  $\text{Cir}(M_a, M_aH_b) \cap xM_a y = B, C$ ;  $\text{Cir}(O, OB) = \text{Cir}(K)$ ;  $\text{Cir}(K) \cap CH_b = A$ . The perpendicular at the point  $M_a$  to the line  $M_aO$  we call the line  $xM_a y$ ; the circle with center  $M_a$  and radius  $M_aH_b$  will meet the line  $xM_a y$  at points  $B$  and  $C$ ; the circle with center  $O$  and radius  $OB$  is called circle  $K$ ; circle  $K$  will meet the line  $CH_b$  at point  $A$ .

**PROBLEM 103.** Given points  $M_a, H_b, H$ .

*Solution.*  $\perp(M_a, H_bH) \cap H_bH = Q$ ;  $2(H_bQ) = B$ ;  $2(BM_a) = C$ ;  $\perp(B, CH) \cap CH_b = A$ . The perpendicular from  $M_a$  to the line  $H_bH$  will meet the line  $H_bH$  at the point  $Q$  (that is,  $Q$  is the orthogonal projection of  $M_a$  to line  $H_bH$ ); extend segment  $H_bQ$  to double its length to get point  $B$ ; double the segment  $BM_a$  to get point  $C$ ; the perpendicular from point  $B$  to line  $CH$  will meet line  $CH_b$  at point  $A$ .

## A Papal Conclave: Testing the Plausibility of a Historical Account

ANTHONY LO BELLO

*Allegheny College*

*Meadville, PA 16335*

Among the stories to be found in Valérie Pirie's *The Triple Crown*, a gossipy and unreliable account of the Papal conclaves since 1458, is one having to do with the election of 1513 that appears particularly suspicious to a reader with some mathematical abilities.

Twenty-five cardinals entered the conclave. The absence of the French element left practically only two contending parties—the young and the old. The former had secretly settled on Giovanni de' Medici; the second openly supported S. Giorgio, England's candidate. . . . The Sacred College had been assembled almost a week before the first serious scrutiny took place. Many of the cardinals, wishing to temporise and conceal their real intentions, had voted for the man they considered least likely to have any supporters. As luck would have it, thirteen prelates had selected the same outsider, with the result that they all but elected Arborenses, the