The Lost Cousin of the Fundamental Theorem of Algebra

TIMO TOSSAVAINEN
Department of Teacher Education
Joensuu University
Savonlinna, Finland
timo.tossavainen@joensuu.fi

Have you ever reflected on the mystery of learning mathematics when you are struggling with a new concept or theorem, and suddenly, it may show up from a new angle and reveal something familiar that you can relate to? Thereafter, the pieces of an unfinished puzzle usually fall nicely into place. This is a story of a couple of such moments and of conquering an intriguing theorem on exponential functions.

The other day my elderly colleague asked me to check whether a lemma, which we would need elsewhere, is true in certain cases. The lemma was the following one.

LEMMA. *For $n \geq 2$, let $\kappa_1 > \cdots > \kappa_n > 0$ and $t_1 > \cdots > t_{n-1} \geq 0$ and suppose that $a_1, \ldots, a_n$ are real numbers with $a_1 > 0$. If the function*

$$f(t) = \sum_{j=1}^{n} a_j \kappa_j^t$$

satisfies $f(t_1) = \cdots = f(t_{n-1}) = 0$, then $f(t) > 0$ for all $t > t_1$.

The lemma seemed to be just another technical proposition, one of many, belonging to the folklore of real analysis. But since I have taken some courses in real analysis, naturally I accepted the challenge.

Rather soon I was able to prove it for $n = 2, 3, 4$, the cases that were the most interesting with respect to our linear algebraic research. And since it was only a technical lemma, I am a little embarrassed to say it now, I just thought to leave it at that.

But the lemma did not leave me in peace. I had a constant feeling that I hadn't yet figured out the deepest essence of the lemma. So, I had to return to it and find out what could have escaped my notice. Sooner or later, something made me think of the fundamental theorem of algebra and its well-known consequence, namely, that a polynomial of degree n has at most n roots. Then, almost immediately, the lemma started to take another shape. And now, if you think carefully enough, you certainly notice that it is very closely related to the following conjecture.

CONJECTURE. *For $n \in \mathbb{N}$ and $j = 0, \ldots, n$, let $0 < \kappa_0 < \cdots < \kappa_n$ and $a_j \in \mathbb{R}$ so that $a_n \neq 0$. Then the function $f : \mathbb{R} \to \mathbb{R}$,*

$$f(t) = \sum_{j=0}^{n} a_j \kappa_j^t$$

has at most n zeros.

There it was! A statement that is very similar to the famous result on the number of the roots of a polynomial. Recall that a polynomial of degree n is a sum of n power functions and a constant function, possibly with some coefficients equal to zero. In other words, the similarity lies in the relation between the number of the terms in the sum and the upper bound for the number of the zeros.

At this point, I remembered that I had proved the lemma only for $n = 2, 3, 4$, not for all $n \geq 2$. The conjecture would not be of any interest if it held only for a few values of n. How should I proceed now?

The first thing that came to my mind was to ask whether a sum of $n + 1$ exponential functions could be presented, similarly as a polynomial of degree n, as a product of at most n simple terms whose zeros are easily detected. To my disappointment, I was not able to find such a presentation. My second thought then was to try to construct an example of a sum of $n + 1$ exponential functions that has exactly n zeros; or even more which would mean the failure of the conjecture.

A method for constructing examples. Now, clearly, an example to be constructed would be most useful if it applied for all $n \geq 1$. Hence I decided to try a recursive method.

Soon I noticed that I have to overcome at least two problems. First, suppose that

$$f_k(t) = a_0\kappa_0^t + \cdots + a_k\kappa_k^t$$

has exactly l zeros $t_1 < \cdots < t_l$ which we may know or not. Now, adding $a_{k+1}\kappa_{k+1}^t$ to f_k implies that $f_{k+1}(t_j) = 0$ does not hold anymore for $j = 1, \ldots, l$. But on the other hand, if $|a_{k+1}\kappa_{k+1}^t|$ were small enough on the interval $[t_1, t_l]$, then the number of the zeros might be controlled somehow when shifting from f_k to f_{k+1}. However, this approach requires that we know the interval where the zeros of f_k lie.

The second problem is that usually we do not know the zeros of f_k and thus the interval $[t_1, t_l]$. This is simply due to the fact that, in most cases, we are not able to find a complete solution to the equation $f_k(t) = 0$. Fortunately, it is possible to gain some information on the zeros of f_k simply by exploring how f_k changes sign. Recall that f_k is continuous for every $k \geq 0$.

For the sake of simplicity, I also decided to use only exponential functions whose bases are at least one. Here is what I did.

For $1 \leq \kappa_0 < \kappa_1$, it is easy to find constants $a_0, a_1 \in \mathbb{R}$ so that

$$f_1(t) = a_0\kappa_0^t + a_1\kappa_1^t$$

has exactly one zero. Suppose then that

$$f_k(t) = a_0\kappa_0^t + \cdots + a_k\kappa_k^t,$$

where $1 \leq \kappa_0 < \cdots < \kappa_k$, has at least k zeros $t_1 < \cdots < t_k$ whose existence has been verified by observing that, for some $0 \leq \delta_0 < \cdots < \delta_k$,

$$f_k(\delta_j) = (-1)^j y_j, \tag{1}$$

in which all y_j's have the same sign and

$$b_k = \min\{|y_j| : j = 0, 1, \ldots, k\} > 0.$$

If we now fix $\kappa_{k+1} > \kappa_k$ and choose a_{k+1} so that a_k and a_{k+1} have different signs and

$$|a_{k+1}| \leq \frac{b_k}{2\kappa_{k+1}^{\delta_k}},$$

then, by the triangle inequality and the fact that κ_{k+1}^t is increasing, $f_{k+1}(\delta_j)$ and $f_k(\delta_j)$ have the same sign and

$$|f_{k+1}(\delta_j)| \geq \frac{b_k}{2} \tag{2}$$

for every $j = 1, \dots, k$. This means that, similar to (1), also $f_{k+1}(\delta_j)$'s oscillate about the t-axis but never equal zero. Hence f_{k+1} has at least k zeros in $[\delta_0, \delta_k]$. Moreover, since κ_{k+1}^t is not bounded, there exists $\delta_{k+1} > \delta_k$ so that

$$\frac{f_{k+1}(\delta_{k+1})}{f_{k+1}(\delta_k)} \le -1. \tag{3}$$

This implies that f_{k+1} has another zero in $[\delta_k, \delta_{k+1}]$. So we conclude that f_{k+1} must have at least $k+1$ zeros in $[\delta_0, \delta_{k+1}]$ altogether. Further, (2) and (3) guarantee that we can proceed with adding new exponential functions as many times as we want to. Here is an explicit example.

EXAMPLE. For any natural number n and $t \in \mathbb{R}$, let

$$f_n(t) = 1^t - \frac{2^t}{2} + \frac{4^t}{4^2} - \frac{8^t}{8^3} + \cdots + (-1)^n 2^{n(t-n)} = \sum_{j=0}^{n} (-1)^j 2^{j(t-j)}.$$

I claim that f_n alternates sign at the sequence $t = 0, 2, 4, \dots, 2n$. Let us verify this for all natural numbers n. For $t = 0$, the leading term is 1^t and it dominates the sum of the absolute values of the other terms. Similarly, for $t = 2n$ the last term dominates. For any even integer $t = 2k$ with $1 \le k \le n-1$, the three terms $j = k-1, k, k+1$ sum to zero. The terms for the leading tail, that is the terms $j = 0, \dots, k-2$ (if any exist), have the sum of their absolute values dominated by the term $j = k-2$. Likewise the terms for the trailing tail, the terms $j = k+2, \dots, n$ (if any exist), have the sum of their absolute values dominated by the term $j = k+2$. Thus the sum of the two tails has the same sign as the term $j = k$, specifically $(-1)^k$, because the terms $j = k-2, k, k+2$ all have the same sign.

For instance, for $n = 5$, we have

$$f_5(t) = 1^t - \frac{2^t}{2} + \frac{4^t}{4^2} - \frac{8^t}{8^3} + \frac{16^t}{16^4} - \frac{32^t}{32^5}$$

and

$$\begin{aligned}
f_5(0) &= 1 - 2^{-1} + 2^{-4} - 2^{-9} + 2^{-16} - 2^{-25} > 0,\\
f_5(2) &= 1 - 2^{1} + 2^{0} - 2^{-3} + 2^{-8} - 2^{-15} < 0,\\
f_5(4) &= 1 - 2^{3} + 2^{4} - 2^{3} + 2^{0} - 2^{-5} > 0,\\
f_5(6) &= 1 - 2^{5} + 2^{8} - 2^{9} + 2^{8} - 2^{5} < 0,\\
f_5(8) &= 1 - 2^{7} + 2^{12} - 2^{15} + 2^{16} - 2^{15} > 0\\
f_5(10) &= 1 - 2^{9} + 2^{16} - 2^{21} + 2^{-24} - 2^{25} < 0.
\end{aligned}$$

Clearly, f_5 has at least 5 zeros.

All in all, I was able to construct a sum of $n+1$ exponential functions that has at least n zeros but not able to confirm that it had no more than n zeros. An important conclusion follows: if the conjecture is true, it is sharp in the sense that the upper bound for the number of zeros is the best possible.

A well-established example often leads the way to the proof. It is nearly a law of nature in mathematics that a well-motivated example fitting a theorem is half the battle of proving the theorem. Therefore, I decided to revise the above construction once again.

First I noticed that the recursive method I had chosen is based on the idea of mathematical induction. Hence, if I could verify the statement of the conjecture in the case $n = k + 1$ using the same statement for $n = k$, the proof would be essentially done. Moreover, whenever I had not been able to count the number of the zeros directly, I had managed by counting something else that is connected to the zeros. And now remember what Rolle's theorem implies. If you know the number of the zeros of the derivative of a differentiable function, then you can bound the number of the zeros of the original function. And again, almost miraculously, the above construction began to change into the argument that converts our conjecture into a theorem!

And now afterward, when I reflect on the proof, I still find it very elegant. The elegance, in my opinion, comes from the fact that it relies only on a few very basic results of classical real analysis and still it reveals quite an interesting property of exponential functions.

To my surprise, I have not been able to find this result in the literature. However, I imagine that several mathematicians may have noticed it in the course of history, and yet we do not know if any actually did. Anyway, I take the liberty to name this theorem and call it *the lost cousin of the fundamental theorem of algebra* due to an apparent resemblance between that famous result and our theorem.

The proof. Let us consider first the case $n = 1$. By writing $\lambda_1 = \kappa_1/\kappa_0$, we have

$$f(t) = \kappa_0^t\big(a_1\lambda_1^t + a_0\big) = \kappa_0^t g(t),$$

where $f(t) = 0$ if and only if $g(t) = 0$. Since $\lambda_1 > 1$ and $a_1 \neq 0$, there is at most one value $t = t_1$ such that $g(t_1) = 0$.

Assume then that the claim holds for some $n = k \geq 1$. Similar to the above, we denote $\lambda_j = \kappa_j/\kappa_0$ in order to have $f(t) = \kappa_{k+1}^t g(t)$, where

$$g(t) = \sum_{j=1}^{k+1} a_j\lambda_j^t + a_0,$$

and $\lambda_{k+1} > \cdots > \lambda_1 > 1$. Again, $f(t) = 0$ if and only if $g(t) = 0$.

Now, for all $t \in \mathbb{R}$, the derivative of g is

$$g'(t) = \sum_{j=1}^{k+1} a_j(\ln\lambda_j)\lambda_j^t = \sum_{j=0}^{k} \beta_j\mu_j^t,$$

where $\mu_j = \lambda_{j+1}$, $\beta_j = a_{j+1}\ln\lambda_{j+1}$ and $\beta_k \neq 0$. By the assumption, there exist at most k distinct numbers $\delta_1 < \cdots < \delta_k$ such that $g'(\delta_1) = \cdots = g'(\delta_k) = 0$. Thus, by Rolle's theorem, there are at most $k + 1$ distinct numbers $t_1 < \cdots < t_{k+1}$ such that $g(t_1) = \cdots = g(t_{k+1}) = 0$. The theorem follows. ■

There is still one thing to tell. My elderly colleague, who seems to know me quite well, reminds me every now and then that after having proved a theorem I should always check whether there is another one around the corner. So, let us consider the following question. Since exponential and logarithm functions share many important analytical properties (all of them are continuous, differentiable, integrable and, except for the constant function, strictly monotone etc.), do we find another lost cousin by replacing the exponential functions with logarithm functions (to different bases) in our theorem?

Well, the logarithm functions have certain arithmetical properties that eventually forces us to answer the question with "No". Let us consider the sum

$$g_n(t) = \sum_{j=0}^{n} a_j \log_{\kappa_j} t,$$

where $1 < \kappa_0 < \kappa_1 < \cdots < \kappa_n$ and a_j's are real numbers so that $a_n \neq 0$. Changing logarithms to the same base e gives us

$$g_n(t) = \sum_{j=0}^{n} a_j \frac{\ln t}{\ln \kappa_j} = \ln t^{\alpha},$$

where

$$\alpha = \sum_{j=0}^{n} \frac{a_j}{\ln \kappa_j}.$$

Now, depending on whether α is greater than, less than, or equal to zero, g_n is, respectively, strictly increasing with $g_n(1) = 0$, strictly decreasing with $g_n(1) = 0$, or $g_n(t) = 0$ for every $t > 0$.

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Not Mixing Is Just as Cool

SAM NORTHSHIELD
SUNY Plattsburgh
Plattsburgh, NY 12901

Newton's law of cooling is a staple of the Calculus curriculum; it is usually presented as a first or second example of a separable differential equation. In that context, the law states that the rate of change of the temperature T of, say, a quantity of fluid is proportional to the difference between the fluid's temperature and the ambient temperature T_∞:

$$\frac{dT}{dt} = -k(T - T_\infty). \tag{1}$$

This is easily solved (part of the difficulty in solving it is dealing with initial conditions):

$$T(t) = T_\infty + (T_0 - T_\infty)e^{-kt} \tag{2}$$

where $T_0 := T(0)$ is the temperature at time $t = 0$.

The following problem is, for many students, a challenging application of Newton's law even given the formula (2).