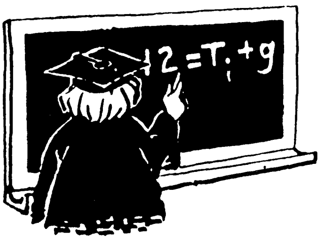

CLASSROOM CAPSULES

Edited by
Warren Page

Classroom Capsules serves to convey new insights on familiar topics and to enhance pedagogy through shared teaching experiences. Its format consists primarily of readily understood mathematics capsules which make their impact quickly and effectively. Such tidbits should be nurtured, cultivated, and presented for the benefit of your colleagues elsewhere. Queries, when available, will round out the column and serve to open further dialog on specific items of reader concern.

Readers are invited to submit material for consideration to:

Warren Page
Department of Mathematics
New York City Technical College
300 Jay Street
Brooklyn, N.Y. 11201

On Partitioning a Real Number

William Staton, University of Mississippi, University, MS

In the most recent of his elegant volumes, Ross Honsberger [Mathematical Gems III, MAA Dolciani Mathematical Exposition Series, 1985, p. 10] proposes, and indicates the solution to, the following problem.

Question 1. How may the positive integer $n > 1$ be partitioned into a sum of positive integers $m_1 + m_2 + \cdots + m_k$ in such a way that the product $m_1 m_2 \cdots m_k$ is as large as possible?

The most common first guess seems to be that most m_i's should be 2. This clearly won't do, however, since three such 2's can be more efficiently replaced by two 3's. Certainly no $m_i = 1$. If some $m_i \geq 4$, then $2(m_i - 2) = (m_i - 4) + m_i \geq m_i$; this rules out summands larger than four, and allows us to replace any 4 by two 2's. Hence, the largest product is obtained by using only 2's and 3's, with at most two 2's.

After announcing this easy but surprising solution, Honsberger wonders why 3 is the key. "Is it because 3 is the integer nearest e?" he asks. It will be shown here, by means of a very natural generalization of the problem, that nearness to e is indeed the key.

Question 2. How may the positive real number r be partitioned into a sum $r = r_1 + r_2 + \cdots + r_k$ of positive real numbers in such a way that the product $r_1 r_2 \cdots r_k$ is as large as possible?

It is not clear that knowing the answer to Question 1 helps the attack on Question 2, but the following easy observation does help: all r_i must be equal. For if $r_i = \alpha \neq \beta = r_j$, then $\alpha\beta < (\alpha+\beta)^2/4$, and a more efficient partition can be gotten by replacing r_i and r_j by $(\alpha+\beta)/2$.

Knowing that all partition parts must be equal allows us to rephrase Question 2 as follows.

Question 3. Given the positive real number r, what is the positive integer k such that $(r/k)^k$ is as large as possible?

Question 3 is completely equivalent to Question 2, but its formulation suggests a generalization not allowed by the language of Question 2. In particular, why should k be forced to take on integer values?

Question 4. Given the positive real number r, what is the positive real number x such that $(r/x)^x$ is as large as possible?

Now, Question 4 may be answered easily by techniques of elementary calculus. Set $y = (r/x)^x$ for $x > 0$, where $r > 0$ is fixed. Use logarithmic differentiation to obtain $y' = (r/x)^x(\ln(r/x) - 1)$, which shows that y is increasing for $0 < x < r/e$, decreasing for $x > r/e$, and maximum for $x = r/e$. Hence, for fixed r, the maximum value of $(r/x)^x$ is $e^{r/e}$. If one chooses, in the setting of Question 4, to retain the partition language of the earlier questions, then one may say that in the most efficient partition, each "part" should be equal to e.

Returning to Question 3 (= Question 2), we note that the behaviour of the function y near r/e allows only two possible choices for k. Of course, in the happy instance where r/e is an integer, the answer to Question 3 is r/e. When r/e is not integral, one of the two nearest integers surrounding r/e must be the optimal value of k. In any case, nearness of k to r/e translates into nearness of r/k to e. In other words, the pieces of the partition should be close to e, as Honsberger suggested.

Let us reformulate our solution to Question 4.

Theorem. *If x and y are positive real numbers whose product is r, then the maximum value of y^x is $e^{r/e}$.*

Note that this theorem provides a quick solution to an old favorite recreational problem: Which is larger, e^π or π^e? One solution is to let $r = \pi e$ and apply the Theorem. The larger number is e^π. For other solutions to this problem, see R. Honsberger's *Mathematical Morsels*, MAA, 1978, or E. Just and N. Schaumberger's "Two More Proofs of a Familiar Inequality" [TYCMJ 6 (May 1975) 45].

Finally, we mention that the monotone nature of $y = (r/x)^x$ on the intervals $(0, r/e)$ and $(r/e, \infty)$ implies that for $c < d$ we have $c^d < d^c$ if $d < e$, and $c^d > d^c$ if $e < c$. For another recent proof of this fact, see J. Rosendahl and J. Gilmore's "Comparing B^A and A^B for $A > B$" [CMJ 18 (January 1987) 50].

———o———

The Relationship Between Hyperbolic and Exponential Functions

Roger B. Nelsen, Lewis and Clark College, Portland, OR

In most calculus texts, therefore presumably in most calculus courses, hyperbolic functions are defined in terms of exponential functions: $\cosh\theta = (e^\theta + e^{-\theta})/2$ and $\sinh\theta = (e^\theta - e^{-\theta})/2$. Then certain identities are verified, and the source of the name "hyperbolic" is revealed: the points $(\cosh\theta, \sinh\theta)$ lie on the right-hand branch of the unit hyperbola $x^2 - y^2 = 1$. What seems to be unjustified or lacking here is a rationale for choosing these particular combinations of exponential functions for defining $\cosh\theta$ and $\sinh\theta$.