Guess a Number—with Lying

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What's green, hangs on a wall, and whistles?
I don't know.
A red herring.
But it's not green.
You can paint it green.
But it doesn't hang on the wall.
You can hang it on the wall.
But it doesn't whistle.
So, it doesn't whistle.

Stainislas Ulam, in his autobiography *Adventures of a Mathematician*, raises an interesting question (italics ours):

> Someone thinks of a number between one and one million (which is just less than 2^{20}). Another person is allowed to ask up to twenty questions, to each of which the first person is supposed to answer only yes or no. Obviously the number can be guessed by asking first: Is the number in the first half-million? and then again reduce the reservoir of numbers in the next question by one-half, and so on. Finally the number is obtained in less than $\log_2(1000000)$. *Now suppose one were allowed to lie once or twice, then how many questions would one need to get the right answer?*

A number of technical papers **[1]**, **[2]** have explored aspects of this problem. Although we discuss some possible answers to Ulam's question, our main emphasis will be on the actual play of the game. We shall assume the first player (called the **Responder**) is allowed to lie at most once. The Responder selects a number x between 1 and n and the second player (the **Questioner**) is allowed k questions. All questions must be of the form: Is $x > a$? After k rounds the Questioner wins if he knows, with proof, the number x. We further allow the Responder to play a "Devil's stategy." By this we mean that the Responder does not actually think of a number x before the game begins but only responds in a consistent manner, that is, at all times there is some x for which he has lied at most once. The reader is urged to try a few games before proceeding. (The values $n = 100$, $k = 11$ make for interesting play.)

We observe that when no lies are permitted the game has an exact solution. If $n \leqslant 2^k$ then the Questioner has a win by the standard halving strategy. If $n > 2^k$ then the Responder has the Devil's strategy (that clever ninth graders occasionally discover) of answering each question so that the reservoir of numbers is at least half of what it was. After k questions there will be more than $n2^{-k}$, hence at least two, numbers remaining.

Let us discuss some possible strategies for the Questioner when one lie is allowed. Suppose that with no lies permitted, u questions suffice to determine the answer. With one lie allowed, we may wish to ask each question twice. If two consistent answers to the same question are not given (this occurs at most once) repeating the same question a third time will reveal the truth. With this strategy, $2u + 1$ questions are sufficient to determine the answer. A modification of the strategy, discovered by M. A. Spencer, does substantially better when u is large. Questions are asked as if there were no lies in u/a groups of a questions. After each group of questions, two further questions are asked to confirm the previous answers. If confirmation of previous answers is not received, which occurs at most once, all $(a + 2)$ answers are thrown out. The lie has been exposed and the Questioner continues with the standard halving strategy.

To illustrate this strategy, suppose $n = 1{,}000{,}000$ and $a = 6$. First six rounds: *Is* $x > 500{,}000$? No. *Is* $x > 250{,}000$? Yes. *Is* $x > 375{,}000$? Yes. *Is* $x > 437{,}500$? No. *Is* $x > 406{,}250$? Yes. *Is* $x > 421{,}875$? Yes. These would be followed by: *Is* $x > 421{,}875$? *Is* $x > 437{,}500$? If the answers are Yes followed by No then the Questioner knows that $421{,}875 < x \leqslant 437{,}500$.

The total number of questions required to determine the number x using this strategy is at most approximately $(u/a)(a+2)+(a+2)$. We set $a \sim \sqrt{2u}$ to minimize this expression so that the Questioner requires at most approximately $u + 2\sqrt{2u} + 2$ rounds to determine the number. This strategy, though suboptimal, is extremely simple to implement.

The sample game shown in FIGURE 1 with $n = 100$ and $k = 11$ shall provide a basis for our further discussion. The answer to the fifth question in that game exposes a lie, though at that point we do not know whether the lie is in response to the first or the fifth question. The game is then reduced to finding a number between 46 and 100 (inclusive) and the normal halving strategy described by Ulam produces the number 65 after the final question and answer.

It is this author's personal experience that the above example is typical of actual play. The Responder makes his lie very early. The Questioner attempts to expose a lie by asking precisely the same question more than once. These, however, are observations of psychology and are not reflected in the mathematical analysis of the game.

In order to mathematically analyze the game, and attempt to answer Ulam's query, we shall imitate the analysis of the game with no lies permitted. The key difference is that we define a **possibility** as an ordered pair (x, L) where x is the number chosen and $L, 0 \leqslant L \leqslant k$, is the number of the question to which the Respondent *lies*. (If $L = 0$, the Responder does not lie.) For $2^k < n(k+1)$ (for example, $n = 100$, $k = 10$) there is a Devil's strategy. To each question the answers Yes and No split the possibilities into two disjoint classes. The Responder gives the answer that leaves the larger class. After k rounds there will remain at least two possibilities. But (and this is essential) these possibilities cannot have the same number x, for if the Questioner had determined the number x he would know, by checking previous answers, to which question the Responder had lied.

We illustrate the Devil's strategy with the situation in our sample game for $n = 100$, $k = 11$ when question 3 is asked but not yet answered. The set of possibilities is:

(x, L)	$0 < x \leqslant 25, L = 2$	25 possibilities
	$25 < x \leqslant 50, L \neq 1,2$	250 possibilities
	$50 < x \leqslant 100, L = 1$	50 possibilities

for the total of 325 possibilities. The third question: Is $x > 38$? splits the above set as follows:

NO CLASS		YES CLASS	
$0 < x \leqslant 25, L = 2$	25	$25 < x \leqslant 38, L = 3$	13
$25 < x \leqslant 38, L \neq 1,2,3$	$13 \times 9 = 117$	$38 < x \leqslant 50, L \neq 1,2,3$	$12 \times 9 = 108$
$38 < x \leqslant 50, L = 3$	12	$50 < x \leqslant 100, L = 1$	50
	154		171

and so the proper Devil's strategy is to answer Yes.

The Questioner's strategy is to select a question that will balance the Yes Class and the No Class as evenly as possible. In the above situation if the question "Is $x > 38$?" is adjusted to "Is $x > 39$?", the No Class gains the possibilities $(39, L)$, $L \neq 1,2,3$ and loses $(39,3)$ for a net gain of eight, making the No Class/Yes Class split 162/163. Thus "Is $x > 39$?" is the proper third question.

When it is the Questioner's turn let us call the set of remaining possibilities the **state** and the number of remaining possibilities the **weight** of the state. The typical state may be written in the form $S^iM^jS^m$ where S^i represents i consecutive numbers x which satisfy all answers but one and

1.	Is $x > 50$?	No.
2.	Is $x > 25$?	Yes.
3.	Is $x > 38$?	Yes.
4.	Is $x > 45$?	Yes.
5.	Is $x > 50$?	Yes.
6.	Is $x > 72$?	No.
7.	Is $x > 59$?	Yes.
8.	Is $x > 66$?	No.
9.	Is $x > 63$?	Yes.
10.	Is $x > 65$?	No.
11.	Is $x > 64$?	Yes.

FIGURE 1. **Sample game with $n = 100$, $k = 11$.**

M^j represents j consecutive numbers x which satisfy all answers. We'll call the numbers S^i a side group and the numbers M^j the main group. In the sample game, the initial state M^{100} becomes successively: $M^{50}S^{50}$, $S^{25}M^{25}S^{50}$, $S^{13}M^{12}S^{50}$, $S^7M^5S^{50}$ and, after the fifth response, S^5S^{50}. (For example, after the third response, $\{39,\dots,50\}$ is the main group and $\{26,\dots,38\}$, $\{51,\dots,100\}$ are the side groups.) If there are t questions remaining, the state $S^iM^jS^m$ has weight $w = i + (t+1)j + m$.

With a little practice, the Questioner can rapidly decide (at least within one number) the appropriate question. Let the state be $S^iM^jS^m$ with t questions remaining. Let a_0 lie in the center of the main group. (If lies were not allowed, then "Is $x > a_0$?" would be the proper question.) The side groups force an adjustment of $(m-i)/(t-1)$ and the Questioner should ask "Is $x > a_0 + (m-i)/(t-1)$?". (Note, roughly, that as t decreases, i.e., as the game nears its conclusion, the influence of the side groups becomes stronger.) If the number $a_0 + (m-i)/(t-1)$ is not in the main group then this method does not apply. Let the state be $S^iM^jS^m$ with $m > i$ (the other case being symmetric) and let E be the largest number in the right side group. If w denotes the weight, the Questioner then asks "Is $x > a$?" where $a = E - (w/2) + j$. For example, after the fourth round in our sample game the state is $S^7M^5S^{50}$ with seven questions remaining ($t = 7$). Here $w = 97$, $E = 100$ and the Questioner asks: "Is $x > 57$?". If Yes is the response, then the new state is S^5S^{43}, where the side groups are the integers from 46 to 50 and from 58 to 100. The Questioner should then follow the normal halving strategy (bearing in mind that the median is no longer the average of the extremes).

For which n, k does this "even splitting" strategy lead to a win for Questioner? A precise answer to this question is difficult because it is not always possible to split the set of possibilities evenly. For example, in our sample game, the state after the first question and answer is $M^{50}S^{50}$ and the weight is $w = 50(11) + 50(1) = 600$. The question "Is $x > 28$?" gives a No Class/Yes Class split of 293/307 and the question "Is $x > 29$?" gives a No Class/Yes Class split of 302/298. This leads us to an analysis of how closely the possibilities may be split.

Suppose that $i - 1$ questions have already been asked and answered and that the current state has weight v_{i-1}. A question "Is $x > a$?" will split the v_{i-1} possibilities into a No Class and a Yes Class. Let $f(a)$ be the size of the No Class. The Questioner seeks an a such that $f(a)$ is as close as possible to $v_{i-1}/2$. (For example, in our sample game after two questions had been asked and answered, the Question "Is $x > 38$?" would induce a No Class/Yes Class split of 154/171. There $v_2 = 325$ and $f(38) = 154$.) If the question "Is $x > a-1$?" is changed to "Is $x > a$?" where a is in the main group then the No Class gains the possibilities (a, L), $L \neq 1,\dots,i$ and loses (a, i) for a net gain of $k - i$. In this case $f(a) = f(a-1) + (k-i)$. (In the sample game, $f(39) = 162$.) When a is in a side group, $f(a) = f(a-1) + 1$ and, of course, when a is not in any group, $f(a) = f(a-1)$. Assume $i \leqslant k-1$. The function f satisfies $f(0) \leqslant v_{i-1}/2$ and $f(n) \geqslant v_{i-1}/2$ (why?) and has jumps of at most $k - i$. Hence for some a, $f(a)$ is within $(k-i)/2$ of $v_{i-1}/2$. (This may be considered a discrete version of the Mean Value Theorem. If, for example, a function goes from less than 162.5 to more than 162.5 with jumps of at most 8 then at some point its value is within 4 of 162.5.) The Questioner asks "Is $x > a$?" for that a. *Regardless* of the answer, the new weight v_i satisfies

$$v_i \leqslant v_{i-1}/2 + (k-i)/2. \tag{1}$$

To analyze inequality (1), we set $w_i = v_i 2^{i-k}$ so that

$$w_i \leqslant w_{i-1} + (k-i)2^{i-k-1} \tag{2}$$

and hence

$$w_{k-1} \leqslant w_0 + \sum_{i=0}^{k-1} (k-i)2^{i-k-1}. \tag{3}$$

The term $(k-i)2^{i-k-1}$ in (2) is essentially the effect of the unevenness of the splitting on the ith question. The terms become significant only when $k-i$ is small. Setting $j = k-i$, we obtain

$$\sum_{i=0}^{k-1} (k-i)2^{i-k-1} = \sum_{j=1}^{k} j2^{-j-1}. \tag{4}$$

The right hand side of (4) is bounded by the infinite sum $\Sigma j 2^{-j-1}$ which converges to 1 (a nice exercise!). Let $v_0 = n(k+1)$, the initial number of possibilities. Then $w_0 = n(k+1)2^{-k}$ and from (3) and (4) it follows that

$$w_{k-1} < n(k+1)2^{-k} + 1$$

and thus

$$v_{k-1} = 2w_{k-1} < 2n(k+1)2^{-k} + 2.$$

Assume that $n \leqslant 2^{k-1}/(k+1)$. The Questioner, applying the halving strategy, can assure $v_{k-1} < 3$. With one question remaining, the state is either M, S, or SS. In the first two cases, the number x has already been determined. In the third case, the lie has already been exposed and the number x may be determined with the last question.

Combining these observations, we obtain the following results for this strategy.

(i) If $n \leqslant 2^{k-1}/(k+1)$, the Questioner wins,
(ii) If $n > 2^k/(k+1)$, the Responder wins.

For example, if $n = 100$, the Responder wins with 10 questions and the Questioner wins with 12 questions. What happens if there are 11 questions? A detailed study (using a small computer) of endgames provided an answer. Checking all states with five questions remaining, it was found that if $v_{k-5} < 26$ then the Questioner has a winning strategy. ($S^4M^3S^4$ is the minimal weight state from which the Responder wins.) A modification of our analysis of inequality (1) can be used to show

$$v_{k-5} < v_0 2^{-k+5} + 6.$$

Thus for this case we can improve (i) to

(iii) If $n \leqslant \frac{5}{8}2^k/(k+1)$, the Questioner wins ($k \geqslant 5$).

For any value of n, the formulae (ii), (iii) determine the required number of questions within one. A more detailed endgame study would certainly increase the constant $5/8$ in (iii). But it seems very difficult to determine whether the answer to Ulam's original problem is twenty-five or twenty-six.

References

[1] T. A. Brylawski, The Mathematics of Watergate (An Analysis of a Two-Party Game), Univ. of N. Carolina, 1978.

[2] D. J. Kleitman, A. R. Meyer, R. L. Rivest, J. Spencer, K. Winklmann, Coping with errors in binary search procedures, J. Comput. System Sci., 20 (1980) 396–404.

[3] S. M. Ulam, Adventures of a Mathematician, Scribner, New York, 1976.