For the other integral, since $|\sin x/x| < 1$ for $x > 0$, we get

$$\left|\int_0^R \frac{e^{-xR}\sin x}{x}\,dx\right| \le \int_0^R e^{-xR}\,dx = \frac{1-e^{-R^2}}{R},$$

which also converges to 0 as $R \to \infty$.

We note that these estimates are identical to those used in [**2**] to show that the order of integration makes no difference in the double integral. Thus, we can view this Green's Theorem calculation as a modification of the double integral method.

References

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Partial Fractions by Substitution

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The standard method for finding the partial fraction decomposition for a rational function involves solving a system of linear equations. In this note, we present a quick method for finding the partial fraction decomposition of a rational function in the special case when the denominator is a power of a single linear or irreducible quadratic factor, that is, the denominator is either $(ax+b)^k$ or $(ax^2+bx+c)^k$ with $4ac > b^2$. For example, we note that substituting $t+2$ for x and then expanding the numerator transforms

$$\frac{x^2+4x-3}{(x-2)^3} \quad \text{to} \quad \frac{t^2+8t+9}{t^3}.$$

Since this last expression splits into

$$\frac{1}{t}+\frac{8}{t^2}+\frac{9}{t^3},$$

it follows that our original function has

$$\frac{1}{x-2}+\frac{8}{(x-2)^2}+\frac{9}{(x-2)^3}$$

as its partial fraction decomposition. We observe that the numbers 9, 8, and 1 in the numerators of the decomposition could also have been obtained as the remainders by successive division of x^2+4x-3 by $x-2$. This method was considered by Kung [**4**] in this journal. Our substitution-expansion method avoids such repeated division as well as the usual systems of equations. (For other methods, see for example [**1, 2, 3, 5, 6, 7**].) It also works equally well on improper fractions, eliminating the need for the initial polynomial division. For our discussion of the general problem of this type, we assume that the denominator is monic (that is, $a = 1$), and consider a rational function

$$R(x) = \frac{N(x)}{D(x)}$$

in the linear and irreducible quadratic cases separately.

The linear case, $D(x) = (x + b)^k$. Let $x = t - b$. Then

$$\frac{N(x)}{(x+b)^k} = \frac{G(t)}{t^k},$$

say, and this immediately yields the desired decomposition. The coefficients of G are the coefficients in the numerators of the partial fractions and can be obtained by binomial expansion or as Taylor polynomial coefficients. We will use binomial expansion, although Brenke [**1**] used Taylor expansion coefficients in a more general case than ours (he required only that the denominator of the fraction have no irreducible quadratic factors).

To illustrate our method, we decompose the function in Kung's first example [**4**],

$$\frac{x^4 + 2x^3 - x^2 + 5}{(2x-1)^5}.$$

After factoring out the 2 and letting $x = t + \frac{1}{2}$, straightforward algebra converts this to

$$\frac{1}{32}\left(\frac{1}{t} + \frac{4}{t^2} + \frac{\frac{7}{2}}{t^3} + \frac{1}{t^4} + \frac{\frac{81}{16}}{t^5}\right),$$

so our decomposition is

$$\frac{\frac{1}{16}}{2x-1} + \frac{\frac{1}{2}}{(2x-1)^2} + \frac{\frac{7}{8}}{(2x-1)^3} + \frac{\frac{1}{2}}{(2x-1)^4} + \frac{\frac{81}{16}}{(2x-1)^5}.$$

To see how the substitution method works for an improper fraction with the same denominator type, consider

$$R(x) = \frac{2x^5 - x^3 + x - 4}{(x+2)^3}.$$

Taking $t = x + 2$, we get

$$2t^2 - 20t + 79 - \frac{154}{t} + \frac{149}{t^2} - \frac{62}{t^3},$$

which gives us the decomposition

$$R(x) = 2x^2 - 12x + 47 - \frac{154}{x+2} + \frac{149}{(x+2)^2} - \frac{62}{(x+2)^3}.$$

The irreducible quadratic case, $D(x) = (x^2 + bx + c)^k$. We first complete the square in order to express $D(x)$ in the form $[(x+p)^2+q]^k$. Now make two substitutions, first $t = x + p$ as before, and then $s = t^2 + q$. We illustrate in the following example:

$$R(x) = \frac{4x^5 - 17x^4 + 45x^3 - 58x^2 + 48x - 8}{(x^2 - 2x + 3)^3}.$$

Then after completing the square, we get

$$\frac{4x^5 - 17x^4 + 45x^3 - 58x^2 + 48x - 8}{[(x-1)^2 + 2]^3}.$$

Setting $t = x - 1$ and simplifying, we get

$$\frac{4t^5 + 3t^4 + 17t^3 + 15t^2 + 19t + 14}{(t^2 + 2)^3}.$$

Letting $s = t^2 + 2$ eventually results in

$$\frac{4t+3}{s} + \frac{t+3}{s^2} + \frac{t-4}{s^3},$$

which gives us the decomposition

$$R(x) = \frac{4x-1}{x^2 - 2x + 3} + \frac{x+2}{(x^2 - 2x + 3)^2} + \frac{x-5}{(x^2 - 2x + 3)^3}.$$

We mentioned earlier Brenke's method using Taylor expansion coefficients which applies to rational functions whose denominators have more than one prime factor. Unfortunately, when irreducible quadratic factors are present, the method requires first, complex linear factorization of quadratic factors, and then, that partial fractions be recombined at the end of the process to recover fractions in real polynomial form.

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