A Relation Between the Jacobian and Certain Generalized Wronskians

KENNETH WOLSSON
Fairleigh Dickinson University
Teaneck, NJ 07666

The purpose of this note is to call attention to an interesting identity involving the Jacobian of a set of functions and certain of its generalized Wronskians. This identity provides a compact solution to an interesting geometric problem and has been helpful in extending the work in [1] (see [2], [3] for related results).

Let the first partial derivatives of $\phi = (\phi_1(t), \ldots, \phi_n(t))$ be assumed to exist at a point $t = (t_1, \ldots, t_n)$. Let J denote the Jacobian of ϕ—the determinant of the matrix $[J]$ whose kth row consists of the components of $\partial\phi/\partial t_k$, and let W_j denote the determinant of the matrix constructed from $[J]$ as follows: Its first row consists of the components of ϕ. Below this we append in order, the rows of $[J]$ omitting the jth one. These determinants belong to a class called *generalized Wronskians* of ϕ ([1] p. 74, [2] pp. 129–132, [3] pp. 138–141).

We claim that if $\phi_i(t) \neq 0$,

$$J(\phi_1, \ldots, \phi_n) = \frac{1}{\phi_i} \sum_j \pm \frac{\partial \phi_i}{\partial t_j} W_j, \tag{1}$$

where "$\pm$" denotes sign alteration starting with "$+$."

To prove (1) we construct a matrix $[K]$ as follows: First form a matrix $[M]$ with the components of ϕ as its first row. The next n rows of $[M]$ are those of $[J]$ in order. To complete $[K]$ we prefix $[M]$ with its own ith column. For example, if $n = 3$ and $i = 2$,

$$[K] = \begin{bmatrix} \phi_2 & \phi_1 & \phi_2 & \phi_3 \\ \dfrac{\partial \phi_2}{\partial t_1} & \dfrac{\partial \phi_1}{\partial t_1} & \dfrac{\partial \phi_2}{\partial t_1} & \dfrac{\partial \phi_3}{\partial t_1} \\ \dfrac{\partial \phi_2}{\partial t_2} & \dfrac{\partial \phi_1}{\partial t_2} & \dfrac{\partial \phi_2}{\partial t_2} & \dfrac{\partial \phi_3}{\partial t_2} \\ \dfrac{\partial \phi_2}{\partial t_3} & \dfrac{\partial \phi_1}{\partial t_3} & \dfrac{\partial \phi_2}{\partial t_3} & \dfrac{\partial \phi_3}{\partial t_3} \end{bmatrix}.$$

Equation (1) follows by expanding the determinant of $[K]$ by cofactors of its first column and noting that it must vanish.

The result is unusual in that the undifferentiated ϕ_i's only *seem* to appear on the right side yet must cancel since they are absent from the left. Also (1) implies that the expression on the right is independent of which ϕ_i is used. We can exploit this to solve the problem of finding the direction in which the functions $\psi_1(t), \ldots, \psi_n(t) \in C^1$ have a common derivative at each interior point of their common domain. If in (1) we put $\phi_i = \exp \psi_i$ (hence $\phi_i \neq 0$) and $\overline{W}_j = W_j(\exp \psi_1, \ldots, \exp \psi_n)$, then (1) tells us that $\langle \overline{W}_1, -\overline{W}_2, \ldots, (-1)^{n+1}\overline{W}_n \rangle$ is the required direction in which $d\psi_i/ds = J(\exp \psi_1, \ldots, \exp \psi_n)/\sqrt{\Sigma \overline{W}_j^2}$ $(i = 1, 2, \ldots, n)$. The only exceptions are of course points at which all $\overline{W}_j$ vanish simultaneously.

We can confirm this result for the case $n = 2$. Here the direction sought is $\langle \overline{W}_1, -\overline{W}_2 \rangle$ with

$$\overline{W}_1 = \begin{vmatrix} \exp \psi_1 & \exp \psi_2 \\ \dfrac{\partial}{\partial t_2} \exp \psi_1 & \dfrac{\partial}{\partial t_2} \exp \psi_2 \end{vmatrix} \quad \text{and} \quad \overline{W}_2 = \begin{vmatrix} \exp \psi_1 & \exp \psi_2 \\ \dfrac{\partial}{\partial t_1} \exp \psi_1 & \dfrac{\partial}{\partial t_1} \exp \psi_2 \end{vmatrix},$$

which by computation can be seen to have the same direction as $\langle \partial(\psi_2 - \psi_1)/\partial t_2, -\partial(\psi_2 - \psi_1)/\partial t_1 \rangle$. We recognize this as the tangent direction to the curve along which $\psi_2 - \psi_1$ is constant and hence along which the derivatives of ψ_1 and ψ_2 are equal. The above mentioned exception occurs only at a point where both components of this direction vanish. At such a point the derivative of $\psi_2 - \psi_1$ vanishes and hence the derivatives of ψ_1 and ψ_2 are equal—in *all* directions.

REFERENCES

1. K. Wolsson, Linear dependence of a function set of m variables with vanishing generalized Wronskians, *Linear Algebra and Its Applications* 117 (1989) 73–80.
2. W. J. LeVeque, *Topics in Number Theory*, Addison-Wesley Publishing Co., Reading, MA, 1956.
3. W. Schmidt, Diophantine Approximation, *Lecture Notes in Mathematics*, No. 785, Springer-Verlag, NY, 1980.

A Surprising Application of the Integral $\int_0^T (1 - F(x))\, dx$ to Revenue Projection

ALLAN KROOPNICK
Social Security Administration
Baltimore, MD 21235

The Social Security Administration annually estimates revenue into its trust funds in order to project present and future benefits and compare those benefits to present and future revenues received. The amount of money flowing into the trust funds depends on the taxable maximum T, which was \$51,300 for 1990. That is, the higher the T, the more money is exposed to taxation and hence more flows into the trust funds. For general overviews of the process see [1] and [2]. For a more mathematical treatment see [3].

In this note, we show that the proportion of wages covered by Social Security and exposed to taxation may be represented by the integral $\int_0^T (1 - F(x))\, dx$, where F is the cumulative distribution function for wage and salaried workers and T is some taxable maximum (a similar approach works for self-employed workers; some minor adjustment is necessary, however, for workers with both wages and self-employment earnings). Thus, if f is a probability density function corresponding to F for such workers and the 1990 employer and employee tax rate was 7.65%, the amount of Social Security tax liability would be given by the expression

$$\text{Tax Liability} = 2(.0765) N \int_0^T (1 - F(x))\, dx \tag{1}$$