Simple Proofs for $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ and $\sin x = x \prod_{k=1}^{\infty}\left(1 - \frac{x^2}{k^2\pi^2}\right)$

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Introduction Almost every textbook on complex analysis or on Fourier series contains a proof of Euler's identity

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Furthermore, there are countlessly many proofs of this result that do not rely explicitly upon complex function theory or Fourier analysis. In the references, we have listed some elementary proofs. Here we shall present a short and simple proof that uses only the definitions of π, sin, cos, and exp, and of course the notion of convergence. Moreover, our proof also gives Euler's identity

$$\sin x = x \prod_{k=1}^{\infty}\left(1 - \frac{x^2}{k^2\pi^2}\right).$$

A trigonometric identity Let m and n be positive integers. It follows from the relation

$$\cos(k+1)x + \cos(k-1)x = 2\cos x \cdot \cos kx$$

that there exists a polynomial T_n of degree n such that for all $x \in \mathbb{R}$

$$\cos nx = T_n(\cos x).$$

In particular we have for positive integers k

$$\cos 2kx = T_k(\cos 2x) = T_k(1 - 2\sin^2 x).$$

This, together with the relation

$$\sin(2k+1)x - \sin(2k-1)x = 2\sin x \cdot \cos(2kx),$$

shows that there is a polynomial F_m of degree m such that for all $x \in \mathbb{R}$

$$\sin(2m+1)x = \sin x \cdot F_m(\sin^2 x).$$

Since $\sin(2m+1)\cdot k\pi/(2m+1) = 0$ for $k = 1, 2, \ldots, m$ we conclude that F_m has zeros at $\sin^2 k\pi/(2m+1)$, $k = 1, 2, \ldots, m$. These zeros are distinct, so F_m has no other zeros; thus

$$F_m(y) = F_m(0) \prod_{k=1}^{m}\left(1 - \frac{y}{\sin^2 \frac{k\pi}{2m+1}}\right),$$

and

$$F_m(0) = \lim_{x \to 0} \frac{\sin(2m+1)x}{\sin x} = 2m+1.$$

Therefore we have

$$\sin(2m+1)x = (2m+1)\sin x \prod_{k=1}^{m} \left(1 - \frac{\sin^2 x}{\sin^2 \dfrac{k\pi}{2m+1}}\right);$$

thus

$$\sin x = (2m+1)\sin \frac{x}{2m+1} \prod_{k=1}^{m} \left(1 - \frac{\sin^2 \dfrac{x}{2m+1}}{\sin^2 \dfrac{k\pi}{2m+1}}\right). \tag{1}$$

Comparison of sums and products For all real t we know that $e^t \geq 1 + t$. So if $1 + t > 0$ we see that

$$e^{-t} = \frac{1}{e^t} \leq \frac{1}{1+t}.$$

Let $u < 1$. The choice $t = u/(1-u)$ leads to

$$e^{-u/(1-u)} \leq 1 - u.$$

For every collection of numbers $u_k \in [0, 1)$ we have

$$1 - \sum_k \frac{u_k}{1-u_k} \leq e^{-\Sigma_k u_k/(1-u_k)} \leq \prod_k (1 - u_k) \leq e^{-\Sigma_k u_k} \leq 1. \tag{2}$$

If we have in addition $\Sigma_k u_k < 1$, then we even know that

$$e^{-\Sigma_k u_k} \leq \frac{1}{1 + \sum_k u_k},$$

and consequently

$$\frac{\sum_k u_k}{1 + \sum_k u_k} \leq 1 - \prod_k (1 - u_k) \leq \sum_k \frac{u_k}{1 - u_k}. \tag{3}$$

Proof that $\Sigma_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$. Let m and N be positive integers, let $m > N$, and let

$$u_k = \left(\frac{\sin \dfrac{x}{2m+1}}{\sin \dfrac{k\pi}{2m+1}}\right)^2 \qquad k = 1, \dots, m.$$

Choose x so small that $0 < \Sigma_k u_k < 1$. It follows from (1) that

$$\prod_{k=1}^{m} (1 - u_k) = \frac{\sin x}{(2m+1)\sin \dfrac{x}{2m+1}},$$

and (3) implies that

$$\frac{\sum_k u_k}{1+\sum_k u_k} \leq 1 - \frac{\sin x}{(2m+1)\sin\frac{x}{2m+1}} \leq \sum_k \frac{u_k}{1-u_k}.$$

Divide by x^2 and let x go to zero. After a short computation we obtain:

$$\frac{1}{6}\left\{1-\frac{1}{(2m+1)^2}\right\} = \sum_{k=1}^{m}\left(\frac{1}{(2m+1)\sin\frac{k\pi}{2m+1}}\right)^2,$$

and after a rearrangement

$$\left|\frac{1}{6}-\sum_{k=1}^{N}\left(\frac{1}{(2m+1)\sin\frac{k\pi}{2m+1}}\right)^2\right| = \frac{1}{6(2m+1)^2}+\sum_{k=N+1}^{m}\left(\frac{1}{(2m+1)\sin\frac{k\pi}{2m+1}}\right)^2.$$

For $0 \leq t \leq \frac{\pi}{2}$ we have $\sin t \geq \frac{2}{\pi}t$, hence the right-hand side is less than

$$\frac{1}{6(2m+1)^2}+\sum_{k=N+1}^{m}\frac{1}{(2k)^2} \leq \frac{1}{6(2m+1)^2}+\frac{1}{4N}.$$

Let $m \to \infty$; we arrive at

$$\left|\frac{1}{6}-\sum_{k=1}^{N}\frac{1}{k^2\pi^2}\right| \leq \frac{1}{4N},$$

and this shows that $\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$.

Proof that $\sin x = x\prod_{k=1}^{\infty}(1-x^2/k^2\pi^2)$. We choose m, N and u_k as in the previous section, but now we take x such that $|x| < \frac{1}{4}N\pi$, and such that $\frac{x}{\pi} \notin \mathbb{Z}$. Again (1) implies that

$$\frac{\sin x}{(2m+1)\sin\frac{x}{2m+1}\prod_{k=1}^{N}(1-u_k)} = \prod_{k=N+1}^{m}(1-u_k),$$

thus we obtain from (2)

$$1-\sum_{k=N+1}^{m}\frac{u_k}{1-u_k} \leq \frac{\sin x}{(2m+1)\sin\frac{x}{2m+1}\prod_{k=1}^{N}(1-u_k)} \leq 1. \tag{4}$$

Using again that $\sin t \geq \frac{2}{\pi}t$ for $0 \leq t \leq \frac{\pi}{2}$we see that

$$u_k \leq \left(\frac{(2m+1)\sin\frac{x}{2m+1}}{2k}\right)^2 \leq \left(\frac{x}{2k}\right)^2;$$

thus

$$\frac{u_k}{1-u_k} \leq \frac{x^2}{(2k)^2-x^2}(k>N).$$

Hence

$$\sum_{k=N+1}^{m} \frac{u_k}{1-u_k} \leq \frac{x^2}{2N-|x|}.$$

Thus it follows from (4) that

$$1-\frac{x^2}{2N-|x|} \leq \frac{\sin x}{(2m+1)\sin\frac{x}{2m+1}\prod_{k=1}^{N}(1-u_k)} \leq 1.$$

Let $m \to \infty$. Then

$$1-\frac{x^2}{2N-|x|} \leq \frac{\sin x}{x\prod_{k=1}^{N}\left(1-\frac{x^2}{k^2\pi^2}\right)} \leq 1.$$

Now let $N \to \infty$ and we obtain for $\frac{x}{\pi} \notin \mathbb{Z}$

$$\sin x = x\prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2\pi^2}\right).$$

For $\frac{x}{\pi} \in \mathbb{Z}$ this is of course also true.

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