Bounds on the Perimeter of an Ellipse via Minkowski Sums

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Since the perimeter $L(E)$ of the ellipse $E=\{(x, y): (x^2/a^2)+(y^2/b^2)\leq 1\}$ $(a, b>0)$ is not an elementary function of a and b, it seems natural to estimate $L(E)$ in terms of averages (or means) of a, b, and the perimeter of the unit disk $D=\{(x, y): x^2+y^2\leq 1\}$. One of the earliest such approximations, $L(E)\cong 2\pi$ $(a+b)/2$, can be found in Kepler's notes [see D. H. Lehmer, "Approximations to the Area of an n-dimensional Ellipsoid," *Canadian Journal of Mathematics* 2 (1950) 267–282]. This estimate is plausible since (Figure 1) it averages the circumradius and the inradius of the ellipse.

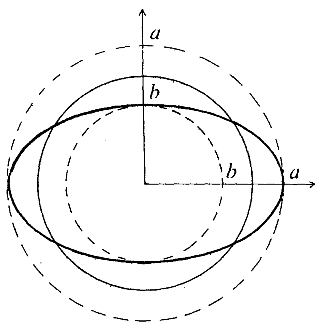

Figure 1.

We will show that $2\pi(a+b)/2$ is a lower bound for $L(E)$. We will also show that the root mean square $M_2=[(a^2+b^2)/2]^{1/2}$ of a and b provides an upper bound, $2\pi M_2$, for $L(E)$. These bounds are well known. [In addition to the above reference, also see M. S. Klamkin's "Elementary Approximations to the Area of n-dimensional Ellipsoids," *American Mathematical Monthly* 78 (1971) 280–283.] However, the geometric proof using the Minkowski sum of two ellipses appears to be new. Our proof that

$$2\pi\left(\frac{a+b}{2}\right)\leq L(E)\leq 2\pi\sqrt{\frac{a^2+b^2}{2}}$$

can also serve as an opportunity for students to learn about convex sets.

Given two plane convex sets P and Q, their Minkowski (or vector) sum is $P+Q=\{p+q: p\in P \text{ and } q\in Q\}$. This is illustrated in Figure 2 for convex polygons P and Q.

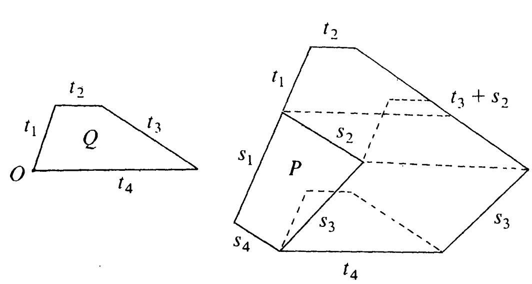

Figure 2.

For a bounded convex set K, the area $A(K)$ and perimeter $L(K) = \sup\{L(P)$: P is a polygon contained in $K\}$ are well defined, non-negative real numbers. [See, for example, I. M. Yaglom and V. G. Boltyanskii's *Convex Figures*, (translated by P. J. Kelly and F. W. Walton) Holt, Rinehart & Winston, New York, 1961.]

It is an interesting property of the Minkowski sum that in general the area of $P + Q$ is *not* equal to the sum of the areas of P and Q. However, the perimeter L *is* additive: $L(P + Q) = L(P) + L(Q)$. For bounded convex sets, it can also be proved that $L(P) \leq L(Q)$ when $P \subseteq Q$. Figure 2 illustrates these properties for convex polygons having side lengths s_i and t_i. The reader should verify the above mentioned properties for the Minkowski sums illustrated in Figures 3a and 3b. Note that the perimeter of a line segment is (by convention) twice its length.

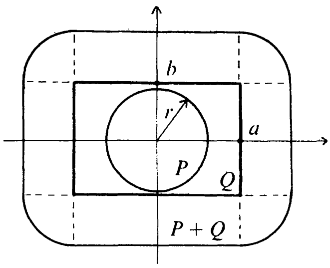

Figure 3a. P is a disk and Q is a rectangle.

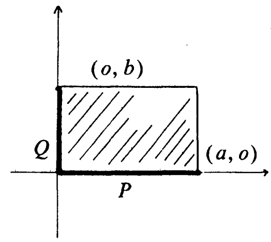

Figure 3b. P and Q are line segments.

For an introduction to convex sets, Minkowski sums, the proofs of the properties above, and applications, see the above referenced (very readable) book by Yaglom and Boltyanskii.

It will be convenient to let rD denote the closed disk $\{(x, y)$: $(x^2 + y^2 \leq r^2\}$ of radius $r > 0$. Our proof that

$$2\pi\left(\frac{a+b}{2}\right) \leq L(E) \leq 2\pi\sqrt{\frac{a^2+b^2}{2}}$$

will follow by establishing that the Minkowski sum of $E = \{(x, y)$: $(x^2/a^2) + (y^2/b^2) \leq 1\}$ and $E' = \{(x, y)$: $(x^2/b^2) + (y^2/a^2) \leq 1\}$ has inradius $a + b$ and circumradius $\sqrt{2(a^2 + b^2)}$, as illustrated in Figure 4 (the dashed lines indicate the boundary of $E + E'$).

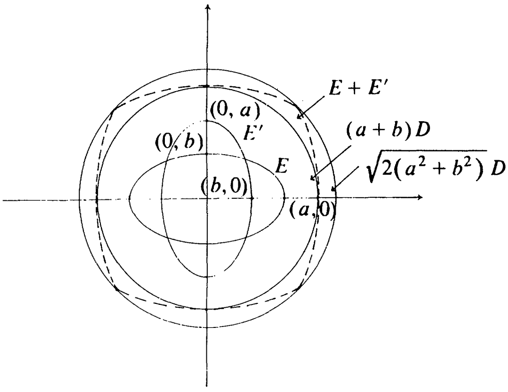

Figure 4.

Theorem. $(a+b)D \subseteq E+E' \subseteq \sqrt{2(a^2+b^2)}\,D$.

Proof. Let $(x, y) \in D$. Then $(a+b)(x, y) = (ax, by) + (bx, ay) \in E + E'$. This proves the first containment.

Now, let $(x, y) \in E$ and $(x', y') \in E'$. Then

$$\begin{aligned}
\|(x, y) + (x', y')\|^2 &= \|(x+x', y+y')\|^2 \\
&= (x+x')^2 + (y+y')^2 \\
&= \left[a\left(\frac{x}{a}\right) + b\left(\frac{x'}{b}\right)\right]^2 + \left[b\left(\frac{y}{b}\right) + a\left(\frac{y'}{a}\right)\right]^2 \\
&\leq (a^2+b^2)\left[\frac{x^2}{a^2} + \frac{x'^2}{b^2}\right] + (b^2+a^2)\left[\frac{y^2}{b^2} + \frac{y'^2}{a^2}\right] \\
&\qquad\qquad \text{(Cauchy-Schwarz Inequality)} \\
&= (a^2+b^2)\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{x'^2}{b^2} + \frac{y'^2}{a^2}\right] \\
&\leq 2(a^2+b^2).
\end{aligned}$$

Hence, $\|(x, y) + (x', y')\| \leq \sqrt{2(a^2+b^2)}$ establishes our second containment. This completes the proof.

It is easy to see that $(a+b)D$ is the largest disk inside $E+E'$. To see that $\sqrt{2(a^2+b^2)}\,D$ is the smallest disk containing $E+E'$, observe that

$$\left(\sqrt{a^2+b^2}, \sqrt{a^2+b^2}\right) = \left(\frac{a^2}{\sqrt{a^2+b^2}}, \frac{b^2}{\sqrt{a^2+b^2}}\right) + \left(\frac{b^2}{\sqrt{a^2+b^2}}, \frac{a^2}{\sqrt{a^2+b^2}}\right) \in E + E'$$

and

$$\left(\sqrt{a^2+b^2}, \sqrt{a^2+b^2}\right) \text{ has length } \sqrt{2(a^2+b^2)}\,.$$

Taking the perimeters of the three sets in our theorem, using the additivity of L and the fact that $L(E) = L(E')$, we obtain

$$2\pi(a+b) \leq 2L(E) < 2\pi\sqrt{2(a^2+b^2)}\,,$$

or,

$$2\pi\left(\frac{a+b}{2}\right) \leq L(E) \leq 2\pi\sqrt{\frac{a^2+b^2}{2}}\,.$$

Equivalent Inequalities

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In this capsule, we show that the following results, usually proved independently, are equivalent: