A New Proof of the Arithmetic-Geometric Mean Inequality

MIHÁLY BENCZE
Braşov, Romania

NORMAN SCHAUMBERGER
Hofstra University
Hempstead, NY 11550

THEOREM. *If* $x_k \geq 0$ *for* $k = 1, 2, \dots, n$ *then we have the following inequality*:

$$x_n^n \geq x_1(2x_2 - x_1)(3x_3 - 2x_2) \cdots (nx_n - (n-1)x_{n-1})$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Proof. We first prove that if $x_k \geq 0$, then

$$x_k^k \geq (kx_k - (k-1)x_{k-1})x_{k-1}^{k-1} \tag{1}$$

with equality if, and only if, $x_k = x_{k-1}$. If $x_k \geq x_{k-1}$, then

$$x_k^{k-1} + x_k^{k-2}x_{k-1} + x_k^{k-3}x_{k-1}^2 + \cdots + x_{k-1}^{k-1} \geq kx_{k-1}^{k-1}$$

and if $x_k \leq x_{k-1}$ the inequality is reversed. In either case

$$x_k^k - x_{k-1}^k \geq kx_{k-1}^{k-1}(x_k - x_{k-1}), \tag{2}$$

with equality if and only if $x_k = x_{k-1}$. Clearly, (1) follows from (2). We now substitute $k = n, n-1, \dots, 2$, successively into (1), to get

$$x_n^n \geq x_1(2x_2 - x_1)(3x_3 - 2x_2) \cdots (nx_n - (n-1)x_{n-1}) \tag{3}$$

with equality if, and only if, $x_k = x_{k-1}$ $(k = 2, \dots, n)$.

Remark. If $a_k \geq 0$ for $k = 1, 2, \dots, n$ then the Arithmetic-Geometric Mean Inequality

$$1/n(a_1 + a_2 + \cdots + a_n) \geq \sqrt[n]{a_1 a_2 \dots a_n} \tag{4}$$

is obtained by substituting $x_k = 1/k(a_1 + a_2 + \cdots + a_k)$ into (3). Equation (4) holds with equality if and only if $a_1 = a_2 = \cdots = a_n$.