but the following easy observation does help: all r_i must be equal. For if $r_i = \alpha \neq \beta = r_j$, then $\alpha\beta < (\alpha+\beta)^2/4$, and a more efficient partition can be gotten by replacing r_i and r_j by $(\alpha+\beta)/2$.

Knowing that all partition parts must be equal allows us to rephrase Question 2 as follows.

Question 3. Given the positive real number r, what is the positive integer k such that $(r/k)^k$ is as large as possible?

Question 3 is completely equivalent to Question 2, but its formulation suggests a generalization not allowed by the language of Question 2. In particular, why should k be forced to take on integer values?

Question 4. Given the positive real number r, what is the positive real number x such that $(r/x)^x$ is as large as possible?

Now, Question 4 may be answered easily by techniques of elementary calculus. Set $y = (r/x)^x$ for $x > 0$, where $r > 0$ is fixed. Use logarithmic differentiation to obtain $y' = (r/x)^x(\ln(r/x) - 1)$, which shows that y is increasing for $0 < x < r/e$, decreasing for $x > r/e$, and maximum for $x = r/e$. Hence, for fixed r, the maximum value of $(r/x)^x$ is $e^{r/e}$. If one chooses, in the setting of Question 4, to retain the partition language of the earlier questions, then one may say that in the most efficient partition, each "part" should be equal to e.

Returning to Question 3 (= Question 2), we note that the behaviour of the function y near r/e allows only two possible choices for k. Of course, in the happy instance where r/e is an integer, the answer to Question 3 is r/e. When r/e is not integral, one of the two nearest integers surrounding r/e must be the optimal value of k. In any case, nearness of k to r/e translates into nearness of r/k to e. In other words, the pieces of the partition should be close to e, as Honsberger suggested.

Let us reformulate our solution to Question 4.

Theorem. *If x and y are positive real numbers whose product is r, then the maximum value of y^x is $e^{r/e}$.*

Note that this theorem provides a quick solution to an old favorite recreational problem: Which is larger, e^π or π^e? One solution is to let $r = \pi e$ and apply the Theorem. The larger number is e^π. For other solutions to this problem, see R. Honsberger's *Mathematical Morsels*, MAA, 1978, or E. Just and N. Schaumberger's "Two More Proofs of a Familiar Inequality" [TYCMJ 6 (May 1975) 45].

Finally, we mention that the monotone nature of $y = (r/x)^x$ on the intervals $(0, r/e)$ and $(r/e, \infty)$ implies that for $c < d$ we have $c^d < d^c$ if $d < e$, and $c^d > d^c$ if $e < c$. For another recent proof of this fact, see J. Rosendahl and J. Gilmore's "Comparing B^A and A^B for $A > B$" [CMJ 18 (January 1987) 50].

The Relationship Between Hyperbolic and Exponential Functions

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In most calculus texts, therefore presumably in most calculus courses, hyperbolic functions are defined in terms of exponential functions: $\cosh\theta = (e^\theta + e^{-\theta})/2$ and $\sinh\theta = (e^\theta - e^{-\theta})/2$. Then certain identities are verified, and the source of the name "hyperbolic" is revealed: the points $(\cosh\theta, \sinh\theta)$ lie on the right-hand branch of the unit hyperbola $x^2 - y^2 = 1$. What seems to be unjustified or lacking here is a rationale for choosing these particular combinations of exponential functions for defining $\cosh\theta$ and $\sinh\theta$.

To answer this, first recall that the circular functions are generally *defined* as coordinates of points on the unit circle. If θ represents the radian measure of the signed angle from the positive x-axis to the radius drawn to a point P on the unit circle, then the coordinates of P are defined to be $(\cos\theta, \sin\theta)$. This is equivalent to denoting by $\theta/2$ the signed area of the circular sector swept out by the radius OP (Figure 1).

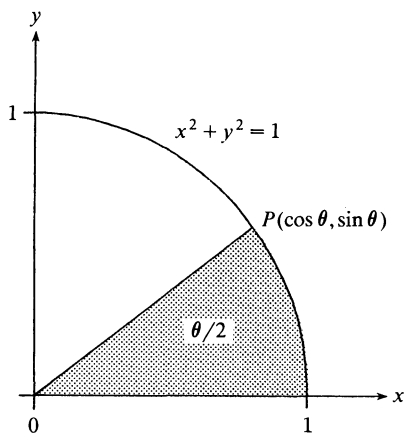

Figure 1.

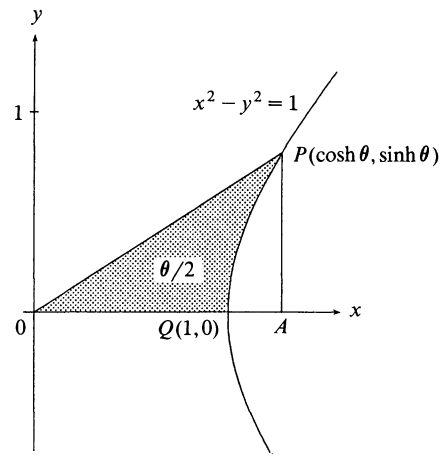

Figure 2.

It is well known that there is an analogous relationship between hyperbolic functions and areas of hyperbolic sectors (see, for example, B. M. Saler's "Inverse Hyperbolic Functions as Areas" [CMJ 16 (March 1985) 128–131]). If $\theta/2$ denotes the signed area of the region swept out by the "radius" OP to the right-hand branch of the hyperbola $x^2-y^2=1$ (the area is taken to be positive when P is in the first quadrant and negative when P is in the fourth quadrant), then the coordinates of P *define* the hyperbolic functions, as in Figure 2.

Now, let us pursue this further. Since we already have $\cosh^2\theta - \sinh^2\theta = 1$, we seek a second equation involving $\cosh\theta$ and $\sinh\theta$. Figure 2 suggests finding an alternate expression for the area $\theta/2$ of the shaded hyperbolic sector, or for the area

$$\frac{1}{2}\cosh\theta\sinh\theta - \frac{1}{2}\theta$$

of the unshaded portion of $\triangle OAP$. As we shall see, a 45° counterclockwise rotation of the shaded region in Figure 2 (or, equivalently, a 45° clockwise rotation of the axes) yields a region whose area can be evaluated easily by use of the natural logarithm function. Since the area of a region is unchanged by such a rotation, this will provide us with a second expression for $\theta/2$ as a natural logarithm function of $\cosh\theta$ and $\sinh\theta$.

With standard results on rotation of axes, the new $\bar{x}-\bar{y}$ coordinates after the 45° rotation are related to the original $x-y$ coordinates by

$$\bar{x} = \frac{\sqrt{2}}{2}(x-y) \qquad \bar{y} = \frac{\sqrt{2}}{2}(x+y).$$

If necessary, this transformation can be easily derived. But perhaps the geometric "proof without words" illustrated in Figure 3 is sufficient at this time, leaving the

general case until later in the course.

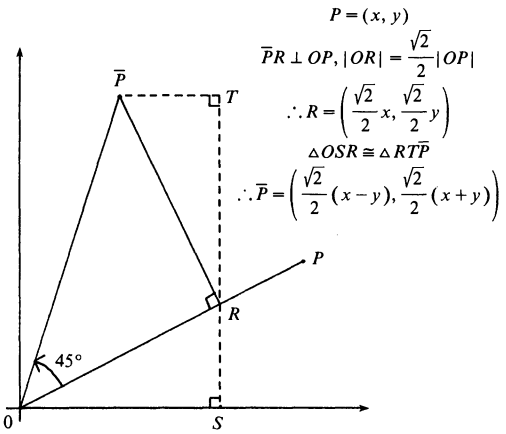

Figure 3.

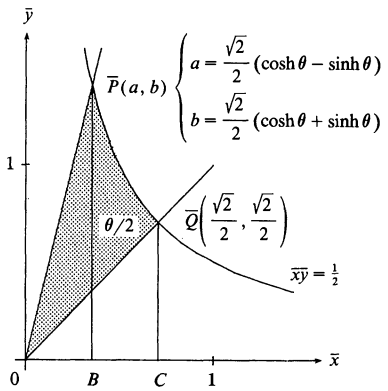

Figure 4.

After the rotation by 45°, we have the situation illustrated in Figure 4. Now the area $\theta/2$ of the shaded sector can be expressed as (replacing $\bar{x}$ by x in the integral)

$$\frac{\theta}{2} = \int_a^{\sqrt{2}/2} \frac{1}{2x}\,dx + \text{area}\,(\triangle O\overline{P}B) - \text{area}\,(\triangle O\overline{Q}C).$$

Since the area of each triangle is $1/4$, the area of the shaded sector is precisely the area under the hyperbola from $\overline{P}$ to $\overline{Q}$, and, hence,

$$\begin{aligned}\theta &= \int_a^{\sqrt{2}/2} \frac{1}{x}\,dx = \ln(\sqrt{2}/2) - \ln(a)\\ &= -\ln(\cosh\theta - \sinh\theta).\end{aligned}$$

Recalling that $(\cosh\theta, \sinh\theta)$ lies on $x^2 - y^2 = 1$, we thus have the system

$$\begin{aligned}\cosh^2\theta - \sinh^2\theta &= 1\\ \cosh\theta - \sinh\theta &= e^{-\theta}.\end{aligned}$$

This has as solution the familiar expressions, $\cosh\theta = (e^{\theta} + e^{-\theta})/2$ and $\sinh\theta = (e^{\theta} - e^{-\theta})/2$, for the hyperbolic cosine and sine.

Another Proof of the Inequality Between Power Means

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The arithmetic-quadratic mean inequality states that for positive numbers $a_1, a_2, \ldots, a_n$:

$$\left(\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}\right)^{1/2} \geq \frac{a_1 + a_2 + \cdots + a_n}{n}, \tag{1}$$

with equality holding if and only if $a_1 = a_2 = \cdots = a_n$. The standard proofs of (1) usually involve considerable algebra or the method of forward and backward induction. [See, for example, N. N. Chentzof, D. O. Shklarsky, and I. M. Yaglom,