The "Ladder Problem"

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A good mathematical problem is like a good song. It keeps coming back time and time again. And just as a song, when it comes back periodically, is still the same song—just sung by someone new in a different style—so a problem often reappears to be proposed in a different framework and/or solved in a different way.

Such is the case of the problem I have come to call, affectionately, the "ladder problem." This problem was originally posed to me by an engineer who said it was being "kicked around" in various engineering circles. I don't know its original source; it may be lost in antiquity.

Before I pose the problem, I would like to ask, "What *is* a 'good' problem?" Of course, that is open to numerous opinions, but surely it would meet several of the criteria listed below and, perhaps, these criteria could be used in evaluating candidates.

1. It can be described quite simply and the crux of the problem can be quickly grasped by one with even minimal mathematical background.
2. It can be nicely cast in a physical, tangible framework.
3. It immediately intrigues and challenges people to whom it is posed.
4. It can be solved in a creative, refreshing and elegant manner, as well as in, perhaps, a more straightforward, but uninteresting way.
5. It can be solved in a variety of ways.
6. It can be solved without using "higher" mathematics.
7. It quickly suggests a variation or generalization which results in an even more challenging problem.

The "ladder problem," I believe, excels in all these categories. Here is the problem as it was posed to me:

> A 16-ft ladder is leaning against a wall in such a manner (FIGURE 1) that one point of the ladder is just touching a box or other obstruction which has a 4-ft by 4-ft cross section and is pushed against the wall. How much of the ladder is between the wall and the point of contact and how much is between the point of contact and the floor?

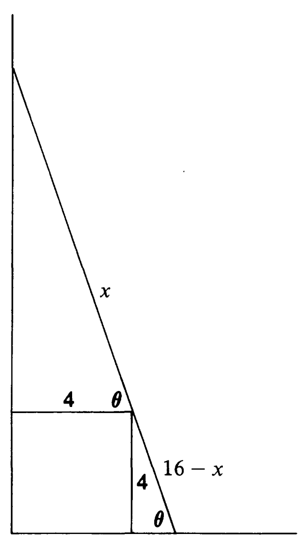

FIGURE 1

"Oh, this looks easy," I thought, as did many others with whom I have shared the problem over the years, and I plunged right in. It didn't take me long to generate a 4th-degree equation—something nearly all would-be solvers quickly do—for by the Pythagorean theorem and similar triangles one can quickly arrive at the proportion

$$4/(16-x)=\sqrt{x^2-16}\,/x$$

which, after squaring both sides, clearing of fractions and simplifying leads to the equation

$$x^4-32x^3+224x^2+512x-4096=0.$$

But I knew that a general 4th-degree equation could only be solved by cumbersome formulas or approximated by some approximation technique. Neither of these appealed to me and my intuition kept insisting there was another way. However, several hours later I laid down my pencil, now several inches shorter, and gave up—at least temporarily.

Soon I, like others after me, took it up again. And again. Eventually (my pride won't allow me to reveal how eventually), I solved it by some tricky trigonometric maneuvers. I raced to show my solution to my colleagues who by now were also "hooked" on the problem. Here is my solution:

(1) From the similar triangles one can quickly obtain the equations

$$\cos\theta=4/x$$
$$\sin\theta=4/(16-x).$$

(2) After solving each of these for x, equating, and a little manipulation, we get the equation

$$\sin\theta+\cos\theta=4\sin\theta\cos\theta.$$

(3) Now square both sides and apply the identities $\sin^2\theta+\cos^2\theta=1$ and $\sin 2\theta=2\sin\theta\cos\theta$. Presto! An equation which is quadratic in $\sin 2\theta$ appears,

$$4\sin^2 2\theta-\sin 2\theta-1=0.$$

(4) Applying the quadratic formula we find that

$$\sin 2\theta=(1+\sqrt{17})/8 \quad \text{(the other root may be discarded)}.$$

Since $0^\circ<\theta<90^\circ$ or $0^\circ<2\theta<180^\circ$, there are two possible solutions to the above equation. They are

$$\theta_1=\sin^{-1}\big((1+\sqrt{17})/8\big)/2\doteq 19.91^\circ$$

and

$$\theta_2=(180-\theta_1)/2\doteq 70.09^\circ.$$

Actually these represent the "same" solution reflecting the fact that the ladder can be positioned in two ways (just switch the similar triangles). These lead to x values of

$$x=4/\cos\theta_1\doteq 4.25$$

and

$$x=4/\cos\theta_2\doteq 11.75.$$

and this shows how the ladder is divided.

My colleagues were sufficiently impressed, and I laid the problem to rest although each year I faithfully resurrected the problem and posed it to my trigonometry class after we had discussed trigonometric identities and equations. I would demonstrate the problem by placing a rectangular wastebasket or a stack of books against a wall and then show them the two ways a meter stick could be positioned so that it touched the floor, the wall, and the wastebasket. For years no one was able to solve it, although several students came up with very accurate approximations. I was disappointed and yet, secretly pleased, for it meant the problem still "belonged" to me. But finally, a student did solve it and to my amazement, without the use of trigonometry.

I swallowed my pride, congratulated him, and settled back to celebrate in the simple beauty of his solution:

(1) He started with the similar triangle proportion mentioned earlier,

$$4/(16-x)=\sqrt{x^2-16}\,/x.$$

(2) Squaring both sides and then making the clever and insightful substitution, $x=8-y$, he transformed a general 4th-degree equation into a special one.

$$16/(8+y)^2=\left((8-y)^2-16\right)/(8-y)^2.$$

(3) Expanding, clearing of fractions and simplifying leads to the equation

$$y^4-160y^2+2048=0,$$

which is quadratic in y^2.

(4) Applying the quadratic equation *twice* produces

$$y=\pm 4\sqrt{5\pm\sqrt{17}}$$

or

$$x=8\pm 4\sqrt{5\pm\sqrt{17}}\,.$$

(5) A little investigation easily shows that the two relevant solutions are

$$8+4\sqrt{5-\sqrt{17}}\doteq 11.75$$

and

$$8-4\sqrt{5-\sqrt{17}}\doteq 4.25.$$

Never one to be outdone, I continued to look for a "better" solution. I have never found one, but very recently I did discover *another* one. I started by realizing that this problem, stripped of a physical setting could be posed as follows:

> Find the intercepts of a line which passes through (4,4) if the distance between the intercepts is 16. (FIGURE 2)

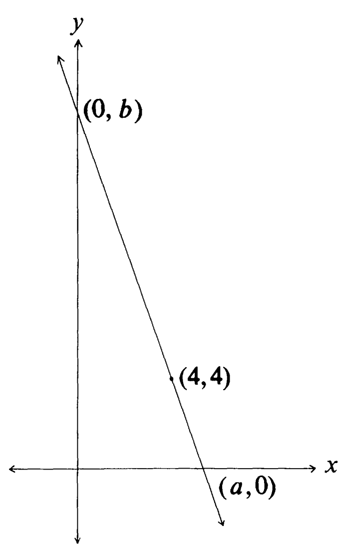

FIGURE 2

Using the same idea of similar triangle proportions one can easily derive the equation $4a+4b=ab$. This equation and the equation $a^2+b^2=16$ form a system which is solvable, although the solution is not pleasant and I won't trouble you with it.

However, this way of looking at the problem did lead me to a realization which is rather fascinating. The same problem can be even more simply described in the following way:

> Find the sides of a rectangle if the area is twice the perimeter and the diagonal is 16. (FIGURE 3).

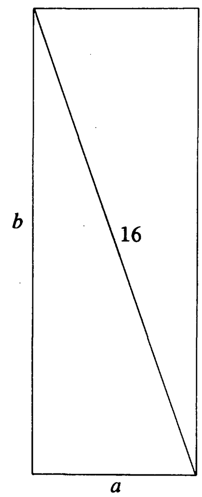

FIGURE 3

This follows immediately from the two equations above if one makes the "ladder" the diagonal of a rectangle with sides of a and b.

As a final note, the alert reader may notice that the solutions presented are possible *only* because of a special characteristic of the obstruction—it has a square cross section. Can anyone solve the problem if the obstruction has, perhaps, a 3-ft by 5-ft cross section? I doubt it—but I've said that before.

Let us remember humility else we be immortalized:

"Young Arthur [Conan Doyle] went to Stonyhurst College in Lancashire...If the name Holmes came from Oliver Wendell, the Sherlock came from a Patrick Sherlock, who was the dullest of all Arthur's contemporaries at Stonyhurst. There were two Moriartys—John Francis and Michael—both of whom won the Stonyhurst prize for mathematics."

Burgess, A., *The Sainted Sleuth*, *Still on the Case*, The New York Times Book Review, January 4, 1987, p. 1.