Three Approaches to a Sequence Problem

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In mathematics we often find that a single problem may be solved by a variety of methods, with each yielding new insights and perspective. In this note we solve a particular problem with three different methods, finding that each method suggests the same generalization (although for varying reasons!). These methods are not new; our goal in collecting them here is to highlight the connections between different techniques.

The focus of our study has appeared (in slightly different forms) in the problem sections of the *American Mathematical Monthly* in 1908 [**3**] and the *College Math Journal* in 1999 [**4**].

DEFINITION. *An integer sequence* $\{x_n\}$ *is* ***prime-divisible*** *if* $p \mid x_p$ *for every prime* p.

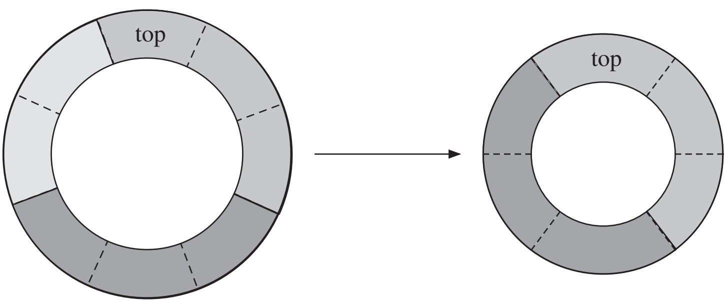
PROBLEM. *Prove that the sequence defined by* $x_1 = 0$, $x_2 = 2$, $x_3 = 3$, *and*

$$x_n = x_{n-2} + x_{n-3} \quad \text{for } n \geq 4$$

is prime-divisible.

Combinatorial argument Our first method relies on the combinatorial maxim of "telling a story" about what the sequence is counting. In the same spirit as Benjamin and Quinn [**1**], we use tilings to interpret the sequence. Consider a circular strip consisting of n equal cells. We wish to tile this strip with pieces that cover two cells (called dominos) and pieces that cover three cells (called triominos). Unlike many other problems of arrangements on a circle, we consider tilings that are related by a rotation to be distinct; in other words, a tiling has a fixed orientation.

The question, quite naturally, is how many tilings are possible for a given n. Denote this count by t_n. We show that, in fact, $t_n = x_n$. It is clear that $t_1 = 0$, $t_2 = 2$ (there are two possible rotational "phases" for the domino), and similarly $t_3 = 3$. Now for $n \geq 4$, pick a particular cell of the circular strip as the "top." For any given tiling, locate the piece covering the top; remove the piece directly beside it (in the counterclockwise direction), and paste the strip back together. What results is an oriented tiling of a smaller strip. One example is shown below.



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If the piece removed was a domino, then what results is a tiling of a $(n-2)$-length strip; if the piece removed was a triomino, then what results is a tiling of a $(n-3)$-length strip. In fact, we get a bijection between the set of tilings of length n and the set of tilings of length $n-2$ or $n-3$. Thus $t_n = t_{n-2} + t_{n-3}$; since $\{t_n\}$ obeys the same initial conditions and recurrence relation as $\{x_n\}$, the two are equal for all n. We have indeed combinatorially represented the sequence in our problem.

Now consider tilings of a prime length p. We prove by contradiction that all of the p rotations of a given tiling are distinct, showing that the total number of tilings must be divisible by p. Suppose there is a tiling such that a rotation by k cells $(0 < k < p)$ leaves the tiling invariant. Since p is prime, k and p are relatively prime; thus there exist integers x and y such that

$$kx + py = 1.$$

Interpreting this physically, if we perform x rotations of the tiling by k cells (under which it is invariant) and y rotations by p cells (again, under which it is invariant), then the result is a rotation by one cell. This shows the tiling is invariant under rotation by a single cell, which is impossible as there are no length 1 pieces. This contradiction completes the proof.

If we examine this argument with an eye for generalization, we see that the absence of any length 1 piece plays the pivotal role. Therefore we do not expect to be able to handle recurrence relations where x_n depends directly on x_{n-1}. We can, however, handle a more general recurrence relation by using a technique found in Benjamin and Quinn [**1**]: we *color* the pieces. Suppose the scenario is the same as above, except there are γ different colors of dominos to use and δ different colors of triominos. By the same proof, the number of ways of tiling a p-length strip will again be divisible by p, and this number is the pth term of the sequence

$$x_1 = 0, x_2 = 2\gamma, x_3 = 3\delta, x_n = \gamma x_{n-2} + \delta x_{n-3} \quad \text{for all } n \geq 4.$$

Thus for any positive integers γ, δ, the sequence above is prime-divisible.

Generating functions Our second method to solve this problem uses the generating function for the series x_n. The generating function for a sequence $\{a_1, a_2, \ldots\}$ is the formally defined series $f(t) = a_1 t^1 + a_2 t^2 + \cdots$. Thus, in our case we are interested in the function

$$f(x) = 0t^1 + 2t^2 + 3t^3 + 2t^4 + 5t^5 + \cdots.$$

The coefficients x_n grow exponentially in n; the ratio test thus shows that for small enough t, $f(t)$ is a well-defined, smooth function. Consider the power series $(t^2 + t^3) f(t)$. Thanks to the recurrence relation, this must match $f(t)$ in all terms of order at least 4. We find that

$$(t^2 + t^3) f(t) = f(t) - 2t^2 - 3t^3,$$

and so

$$f(t) = \frac{2t^2 + 3t^3}{1 - t^2 - t^3} = (-t)\frac{d}{dt}\left[\log(1 - t^2 - t^3)\right].$$

The pth term of the sequence is given from the generating function by $f^{(p)}(0)/p!$. If we take p derivatives of the above product with the Leibniz rule, notice that only one term survives after setting $t = 0$. This term corresponds to applying exactly one derivative to $(-t)$. Thus,

$$x_p = \frac{1}{p!} \cdot \left(p \cdot (-1) \cdot \frac{d^p}{dt^p} \left[\log(1 - t^2 - t^3) \right] \right) \Bigg|_{t=0}$$
$$= (-p) \cdot (\text{coefficient of } t^p \text{ in } \log(1 - t^2 - t^3)).$$

Now using the Taylor series for $\log(1 - t)$, we find that

$$-\log(1 - t^2 - t^3) = (t^2 + t^3) + \frac{(t^2 + t^3)^2}{2} + \frac{(t^2 + t^3)^3}{3} + \cdots.$$

For each $n \geq p$, $(t^2 + t^3)^n$ contributes no t^p term. Thus, the coefficient c of t^p is a sum of fractions, all of whose denominators are less than p. For prime p, this implies that the denominator of c in reduced form is relatively prime with p, so $x_p = -pc$ is indeed divisible by p. This completes the second proof.

As before, we look to generalize this proof technique to other sequences. If the recurrence relation had x_{n-1} dependence, we would have introduced a linear term into the $t^2 + t^3$ factor. Thus we could no longer have argued away the pth term in the log series, which was a key step in the proof. For this reason we do not expect to handle recurrences with x_{n-1} dependence. For the most general remaining recurrence, $x_n = \gamma x_{n-2} + \delta x_{n-3}$, this proof technique still works as long as the numerator of the resulting generating function $f(t)$ is (a constant times) t times the derivative of the denominator. Since the denominator is $1 - \gamma t^2 - \delta t^3$ and the numerator is $x_1 t + x_2 t^2 + x_3 t^3$, prime-divisibility holds for multiples of the initial conditions $x_1 = 0$, $x_2 = -2\gamma$, $x_3 = -3\delta$. Except for a superficial negative sign, this is the same form as the generalization we found in the previous section!

Field-theoretic solution For our third and final solution method, we use some field theoretic ideas. This proof requires slightly more background than the others; we refer the reader to any introductory algebra text, such as Dummit and Foote [2].

Consider the characteristic polynomial

$$f(t) = t^3 - t - 1$$

for the sequence $\{x_n\}$. This has three distinct roots r_1, r_2, and r_3 in the complex plane; thus there exist complex constants A, B, and C such that

$$x_n = Ar_1^n + Br_2^n + Cr_3^n$$

for all n. We show that $A = B = C = 1$.

It suffices to verify equality for $n = 0$, $n = 1$, and $n = 2$, since those initial conditions determine the sequence. (While the problem did not specify a zeroth term of $\{x_n\}$, it is defined uniquely by extending the recurrence relation. Solving $x_3 = x_1 + x_0$, we see that $x_0 = 3$.) Obviously $r_1^0 + r_2^0 + r_3^0 = 3 = x_0$. Further, since

$$f(t) = t^3 - t - 1 = (t - r_1)(t - r_2)(t - r_3),$$

by equating quadratic coefficients we have $r_1^1 + r_2^1 + r_3^1 = 0 = x_1$. Equating linear coefficients, $r_1 r_2 + r_1 r_3 + r_2 r_3 = -1$; thus

$$r_1^2 + r_2^2 + r_3^2 = (r_1 + r_2 + r_3)^2 - 2(r_1 r_2 + r_1 r_3 + r_2 r_3) = 2 = x_2.$$

This shows that, indeed, $x_n = r_1^n + r_2^n + r_3^n$ for all n.

Now fix a prime p. By the Multinomial Theorem for the power of a three-term sum, a generalization of the Binomial Theorem, we have

$$(r_1 + r_2 + r_3)^p = \sum_{\substack{a,b,c\geq 0 \\ a+b+c=p}} \frac{p!}{a!\,b!\,c!} r_1^a r_2^b r_3^c.$$

The coefficient $p!/(a!\,b!\,c!)$ is always an integer and, unless (a, b, c) is some permutation of $(p, 0, 0)$, it is divisible by p. Thus we may write $(r_1 + r_2 + r_3)^p = r_1^p + r_2^p + r_3^p + p \cdot z$, where

$$z = \sum_{\substack{0\leq a,b,c<p \\ a+b+c=p}} \frac{(p-1)!}{a!\,b!\,c!} r_1^a r_2^b r_3^c$$

is some integer-linear combination of products of r_1, r_2, and r_3. Substituting $x_n = r_1^n + r_2^n + r_3^n$,

$$x_1^p = x_p + p \cdot z$$
$$-x_p = p \cdot z.$$

Now z is an algebraic integer; since $p \cdot z$ is an integer, it follows that z is in fact an integer. Thus $p \mid x_p$, completing the proof.

We again attempt to generalize this proof. It was crucial that f have three distinct roots r_1, r_2, r_3, and that the sum $r_1 + r_2 + r_3$ vanish. This restricts us to recurrences of the form $x_n = \gamma x_{n-2} + \delta x_{n-3}$. Now the proof will proceed for any multiple of $x_n = r_1^n + r_2^n + r_3^n$; this gives the initial conditions $x_1 = 0$, $x_2 = 2\gamma$, $x_3 = 3\delta$. We have yet again found the same generalization.

Conclusions Our three methods offer different interpretations of the problem. Depending on one's point of view, the initial conditions that made our sequence prime-divisible arose from

- counting base case tilings of a circular strip;
- matching the numerator of the generating function with the derivative of the denominator; or
- using sums of powers of roots of the characteristic polynomial.

Also notably, the three methods used the condition of primality in slightly different ways.

It is especially compelling that each of these interpretations led to the same natural generalization of the problem, giving initial conditions for $x_n = \gamma x_{n-2} + \delta x_{n-3}$ to be prime-divisible. The fact that this generalization arose thrice suggests that it is actually the "correct" one. In fact, with suitable restrictions on the characteristic polynomial, it does cover all prime-divisible third-order recurrences.

If those restrictions are relaxed, though, there are more families of prime-divisible sequences. We prove this, giving a catalogue of such sequences, in an upcoming companion paper. In the mean time, we invite the reader to use any of these three approaches (or another!) to discover these additional recurrence families for yourself.

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Summary In this note we examine a well-studied problem concerning the terms of a certain linear recurrence modulo prime numbers. We present three solutions to this problem and examine the similarities and differences between them. In particular, despite using primality in different ways, all three proofs yield the same generalization of the original problem.

Isoperimetric Sets of Integers

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The celebrated isoperimetric theorem says that the circle provides the least-perimeter way to enclose a given area. In this note we discuss a generalization which arose at a departmental research seminar [**1**] and which moves the isoperimetric problem from geometry to number theory and combinatorics. Instead of Euclidean space, let's take the set $\mathbb{N}_0$ of nonnegative integers:

$$\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}.$$

For any subset S of $\mathbb{N}_0$, we define volume and perimeter as follows:

$$\begin{aligned}\mathrm{vol}(S) &:= \text{sum of elements of } S\\ \mathrm{per}(S) &:= \text{sum of elements of } S \text{ whose predecessor and successor}\\ &\ \text{are not both in } S.\end{aligned}$$

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