NOTES

Algebraic Set Operations, Multifunctions, and Indefinite Integrals

MILAN V. JOVANOVIĆ
Matematički Institut
Kneza Mihaila 35
Beograd, Yugoslavia

VESELIN M. JUNGIĆ
Simon Fraser University
Burnaby, B.C., Canada

The fact that an indefinite integral is a set of functions is often ignored, perhaps because of the apparent simplicity of the situation. However, if we regard

$$\int \frac{dx}{x} \quad \text{or} \quad \int \frac{\cos x}{\sin x}\, dx$$

as functions, we can easily develop fallacious proofs of such "identities" as $0 = 1$.

In this note we introduce a semigroup operation on the set of all nonempty subsets of a vector space. Then we indicate how the indefinite integral can be viewed as a set-valued function (or *multifunction*) and how this point of view avoids the fallacies mentioned above. Finally, we show how the multifunction given by the indefinite integral induces a linear function on the space of continuous functions.

Algebraic set operations Let X be a vector space over the real numbers, and let $P(X)$ denote the family of all nonempty subsets of X. We define addition and scalar multiplication on the family $P(X)$ by

$$A + B = \{a + b \colon a \in A,\ b \in B\}$$

and

$$\alpha A = \{\alpha a \colon a \in A\},$$

where $A, B \in P(X)$ and $\alpha \in \mathbb{R}$. In particular, $A - B = A + (-1)B$. These are called *algebraic set operations*. Notice that $(P(X), +)$ is not a group if $X \neq \{0\}$. Indeed, $\{0\}$ is the neutral element in $(P(X), +)$, and for every $A \in P(X)$

$$A + X = X,$$

so X has no inverse element. The operation $+$ is associative and commutative. The following properties of the operations hold

$$\alpha(\beta A)=(\alpha\beta)A \tag{1}$$

$$\alpha(A+B)=\alpha A+\alpha B \tag{2}$$

$$1A=A. \tag{3}$$

The inclusion

$$(\alpha+\beta)A\subseteq\alpha A+\beta A \tag{4}$$

holds, but the opposite inclusion need not hold. (Setting $X=\mathbb{R}$, $A=\{-1,1\}$, and $\alpha=\beta=\frac{1}{2}$ gives a counterexample.) Other properties of algebraic set operations include the following, where $A, B, C\in P(X)$ and $\alpha\in\mathbb{R}$:

$$0\in A-A \tag{5}$$

$$(0\in A \text{ and } A+B\subseteq C)\Rightarrow B\subseteq C \tag{6}$$

$$\alpha\neq 0\Rightarrow(A\subseteq B\Leftrightarrow\alpha A\subseteq\alpha B) \tag{7}$$

$$A+B\subseteq C\Rightarrow B\subseteq C-A \tag{8}$$

$$A=B\Rightarrow A+C=B+C. \tag{9}$$

The converse to (8) does not hold, as shown by the example $A=B=X$ and $C=\{0\}$. For $X=\mathbb{R}$, $A=[0,1]$, $B=\{1\}$, and $C=[1,2]$ we have $A+B=C$ and $B\neq C-A$. Thus, in general, $A+B=C$ does not imply $B=C-A$.

Some formulae, that do not hold in the general case, do hold for convex sets. A set $A\in P(X)$ is *convex* if for every α, $\beta\in\mathbb{R}$ such that $\alpha\geq 0$, $\beta\geq 0$, and $\alpha+\beta=1$,

$$\alpha A+\beta A\subseteq A.$$

The converse implication to (9) need not hold in general (e.g., $A\neq X$ and $C=X$). However, if X is a normed vector space, B is closed and convex, and C is bounded, then (see, e.g., [**2**, Lemma 1])

$$A+C\subseteq B+C\Rightarrow A\subseteq B.$$

Let A be a convex set, $\alpha>0$ and $\beta>0$; then

$$\frac{\alpha}{\alpha+\beta}A+\frac{\beta}{\alpha+\beta}A\subseteq A.$$

From (7) and (2) we get $\alpha A+\beta A\subseteq(\alpha+\beta)A$, and by (4),

$$\alpha A+\beta A=(\alpha+\beta)A.$$

In particular, if A is convex then

$$A+A=2A.$$

A subset C of X is a *subspace* if for all $\alpha,\beta\in\mathbb{R}$

$$\alpha C+\beta C\subseteq C.$$

Now, for fixed $x\in X$, the subset $\{x\}+C$ is called an *affine subspace* (*flat*) *parallel to* C. A flat is a convex set. If C is a subspace of X and L is a flat parallel to C, the following algebraic properties are easily proved:

$$C + C = C \tag{10}$$

$$L - L = C \tag{11}$$

$$\alpha \neq 0 \Rightarrow \alpha C = C \tag{12}$$

$$C - C = 0 \tag{13}$$

$$A \subseteq C \Rightarrow A + C = C \tag{14}$$

$$A \subseteq C \Rightarrow A + L = L. \tag{15}$$

For more on algebraic operations with convex sets, see **[3]**.

Indefinite integrals Let $I \subseteq \mathbb{R}$ be an interval, $C(I)$ the vector space of all continuous real functions on I, $C^1(I)$ the subspace of all continuously differentiable functions, and C the subspace of all constant functions. A differentiable function φ is a *primitive function* of f if $\varphi' = f$ holds. The set of all primitive functions of f is called the *indefinite integral* of f, and denoted by

$$\int f = \{\varphi \colon \varphi' = f\}.$$

Let $f, g \in C(I)$ and $\alpha \in \mathbb{R}$; then

$$\int f \neq \phi \tag{16}$$

$$C = \int f - \int f \tag{17}$$

$$\int f + g = \int f + \int g \tag{18}$$

$$0 \cdot \int f \subseteq \int 0 \cdot f \tag{19}$$

$$\varphi \in \int f \Leftrightarrow \int f = \{\varphi\} + C \tag{20}$$

$$\alpha \neq 0 \Rightarrow \int \alpha f = \alpha \int f \tag{21}$$

$$A \subseteq C \Rightarrow \int f + A = \int f \tag{22}$$

$$f \in C^1(I) \Rightarrow \int f' = \{f\} + C \tag{23}$$

$$u, v \in C^1(I) \Rightarrow \int uv' = \{uv\} - \int u'v. \tag{24}$$

We prove the properties (18) and (24); the others have similar proofs. For (18), let $\varphi \in \int f$ and $\psi \in \int g$; then $\int f + \int g = (\{\varphi\} + C) + (\{\psi\} + C)$. By the commutative and associative laws and (10), we have

$$\int f + \int g = \{\varphi + \psi\} + C.$$

Since $(\varphi + \psi)' = f + g$, (20) shows that

$$\int f + \int g = \int (f+g).$$

To deduce (24), the formula for integration by parts, let $\varphi \in \int uv'$. Then $\varphi' = (uv)' - u'v$. Since $\int (uv)' - u'v = \{uv\} - \int u'v$ we have $\varphi \in \{uv\} - \int u'v$, so $\int uv' \subseteq \{uv\} - \int u'v$. Conversely, if $\varphi \in \{uv\} - \int u'v$ there exists a function $\psi \in \int u'v$ such that $\varphi = uv - \psi$. Since $\varphi' = (uv)' - u'v = uv'$, we have $\varphi \in \int uv'$, so $\{uv\} - \int u'v \subseteq \int uv'$.

Note that, by (20), $\int f$ is a flat, so (11) implies (17).

Example 1. Let $I = (0, \pi)$ and for $x \in I$, $f(x) = \cos x / \sin x$. Let $J = \int f$. Using integration by parts, where $u(x) = 1/\sin x$ and $v(x) = \sin x$, we get $J = 1 + J$. Failure to notice that an indefinite integral is a set leads to the fallacious conclusion that $0 = 1$. However, from (24) we have $J = \{uv\} + J$ where $uv \in C$. Therefore, by (22), $J = J$.

A mistake can also be made in calculating integrals when incorrect set formulae are used. For example, from the "equality"

$$J = u(x)v(x) - J$$

one might conclude that $2J = u(x)v(x)$, which is also incorrect.

Example 2. Let $I = \mathbb{R}$, $f(x) = e^x \sin x$, and $g(x) = e^x(\cos x - \sin x)$. Using (24) we get

$$\int f = \{g\} - \int f.$$

Clearly, $\{g\} \neq 2\int f$. However, $\int f = \{g\} - \int f$ implies, by (9), that

$$\int f + \int f = \{g\} + \left(\int f - \int f\right).$$

Therefore

$$2\int f = \{g\} + C,$$

and, by (7) and (12),

$$\int f = \left\{\frac{g}{2}\right\} + C.$$

The antiderivative multifunction Let X and Y be Banach spaces. A multivalued function (or simply a multifunction) $F\colon X \to P(Y)$ is called *convex* if its graph

$$\operatorname{gr} F = \{(x, y) \in X \times Y \colon y \in F(x)\}$$

is a convex set. This is equivalent to th e condition that

$$\alpha F(x_1) + \beta F(x_2) \subseteq F(\alpha x_1 + \beta x_2)$$

for all $x_1, x_2 \in X$, and all $\alpha \geq 0$, $\beta \geq 0$ with $\alpha + \beta = 1$.

We say that F is *closed* on X if $\operatorname{gr} F$ is a closed set in the product topology on $X \times Y$. This is equivalent to the condition that $x_k \to x$, $y_k \to y$, $x_k \in X$, and $y_k \in F(x_k)$ imply $y \in F(x)$.

Le Van Hot [**1**, Theorem 2] has proved that if X and Y are Banach spaces and F: $X \to P(Y)$ is a convex closed multifunction such that $\operatorname{dom}(F) = X$ and $F(x_0)$ is bounded for some $x_0 \in X$, then there exists a unique linear single-valued function T: $X \to Y$ such that

$$F(x) = F(0) + T(x). \tag{25}$$

Without the assumption that $F(x_0)$ is bounded for some $x_0 \in X$, the conclusion of Le Van Hot's Theorem is not true. Consider, for example, the multifunction F: $X \to P(Y)$, given by $F(f) = \int f$, where $X = Y = C([0,1])$. Note that $C([0,1])$ is a Banach space with

$$\|f\| = \max\{|f(x)|: x \in [0,1]\}.$$

By (18) and (21), F is a convex function. Also, F is closed by the uniform convergence and differentiation theorem [**4**, Theorem 7.17]. By (16), we have

$$\operatorname{dom}(F) = \{f \in X: F(f) \neq \phi\} = C([0,1]).$$

However, $F(f)$ is unbounded for each $f \in C([0,1])$. In this case the formula (25) becomes

$$\int f = \int 0 + \{T(f)\} \tag{26}$$

or, equivalently,

$$\int f = \{T(f)\} + C.$$

If we let T: $C([0,1]) \to C([0,1])$ be the linear function given by

$$T(f)(x) = \varphi(x) - \varphi(c)$$

where $\varphi \in \int f$ and c is any number in $[0,1]$, then (26) holds. However, T is not unique.

REFERENCES

1. Le Van Hot, *On the open mapping principle and convex multivalued mappings*, Acta. Univ. Carol.-Math. Phys. 26 (1985), 53–59.
2. H. Rådström, *An embedding theorem for spaces of convex sets*, Proc. Amer. Math. Soc. 3 (1952), 165–169.
3. R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ, 1970.
4. W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, New York, NY, 1976.