A Fundamental Theorem of Calculus that Applies to All Riemann Integrable Functions

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The usual form of the Fundamental Theorem of Calculus is as follows:

THEOREM 1. *Let f be Riemann integrable on* $[a, b]$ *and let g be a function such that* $g'(x) = f(x)$ *on* $[a, b]$. *Then*

$$\int_a^b f(x)\,dx = g(b) - g(a).$$

Unfortunately, this theorem only applies to Riemann integrable functions that are derivatives. Thus it cannot even be used to integrate the following simple function

$$f(x) = \begin{cases} 0 & \text{if } -1 \leqslant x < 0 \\ 1 & \text{if } 0 \leqslant x \leqslant 1. \end{cases}$$

It is the purpose of this note to present a theorem that does apply to every integrable function. In stating our result we will need the following definitions.

Definition 1. The function $f\colon [a, b] \to R$ satisfies a Lipschitz condition if there exists $M > 0$ such that

$$|f(x) - f(y)| \leqslant M|x - y| \quad \text{for all } x \text{ and } y \text{ in } [a, b].$$

Definition 2. A set E of real numbers has measure zero if for each $\varepsilon > 0$ there is a finite or infinite sequence $\{I_n\}$ of open intervals covering E and satisfying $\Sigma_n |I_n| \leqslant \varepsilon$ where $|I_n|$ is the length of I_n. If a property holds *except* on a set of measure zero, it is said to hold almost everywhere.

In **[2]** the author gave an elementary proof of the following result.

LEMMA. *If* $f\colon [a, b] \to R$ *satisfies a Lipschitz condition and* $f'(x) = 0$ *except on a set of measure zero, then f is a constant function on* $[a, b]$.

The proof required no measure theory other than the definition of a set of measure zero. This lemma was then used to prove that a bounded function that is continuous almost everywhere is Riemann integrable. We will use it here to establish our general form of the Fundamental Theorem of Calculus.

THEOREM 2. *Let f be Riemann integrable on* $[a, b]$ *and let g be a function that satisfies a Lipschitz condition and for which* $g'(x) = f(x)$ *almost everywhere. Then*

$$\int_a^b f(x)\,dx = g(b) - g(a).$$

Proof. Let $F(x) = \int_a^x f(t)\,dt$. Since f is bounded, F satisfies a Lipschitz condition. From the fact that f is continuous except on a set of measure zero (see **[3]** for an elementary proof), it follows that $F'(x) = f(x)$ almost everywhere. (This shows that every Riemann integrable function is almost everywhere the derivative of a function satisfying a Lipschitz condition.) It follows at once that

$$(F-g)'(x)=F'(x)-g'(x)=f(x)-f(x)=0$$

almost everywhere. In addition $F-g$ satisfies a Lipschitz condition. By the lemma there exists a real number k such that $F(x)=g(x)+k$ on $[a,b]$. Setting $x=a$ we have $k=-g(a)$. Finally, setting $x=b$, we get

$$\int_a^b f(x)\,dx=F(b)=g(b)-g(a),$$

which completes the proof.

Note that Theorem 2 includes Theorem 1 since any function that has a bounded derivative satisfies a Lipschitz condition.

Let us now integrate the following function. Define

$$f(x)=\begin{cases}-x & \text{if } x\in S=\{1,1/2,1/3,\dots\}\\ x^2+1 & \text{if } x\in[0,1]\setminus S.\end{cases}$$

Since f is bounded and continuous except on $S\cup\{0\}$, a set of measure zero, it is Riemann integrable. Let $g(x)=x^3/3+x$. Then g satisfies a Lipschitz condition and we have that $g'(x)=x^2+1=f(x)$ almost everywhere. Therefore,

$$\int_0^1 f(x)\,dx=g(1)-g(0)=4/3.$$

In this case $g'(x)\neq f(x)$ on an infinite set and yet Theorem 2 can still be used.

In closing, we give a useful corollary of Theorem 2.

Corollary. *Let f be Riemann integrable on $[a,b]$ and let g be a continuous function such that $g'(x)=f(x)$ except on a countable set. Then*

$$\int_a^b f(x)\,dx=g(b)-g(a).$$

Proof. To use Theorem 2 we need only show that g satisfies a Lipschitz condition. Since f is integrable there exists $M>0$ such that $|f(x)|\leqslant M$ for all x in $[a,b]$. Thus $-M\leqslant g'(x)\leqslant M$ except on a countable subset of $[a,b]$. Let $h(x)=Mx-g(x)$. Since h is continuous on $[a,b]$ and $h'(x)=M-g'(x)\geqslant 0$ except on a countable set, it follows from a result in [1] that h is increasing on $[a,b]$. Thus for c and d in $[a,b]$ with $c<d$ we have $h(c)\leqslant h(d)$ which gives $g(d)-g(c)\leqslant M(d-c)$. Similarly, we can show that $-M(d-c)\leqslant g(d)-g(c)$ and therefore $|g(d)-g(c)|\leqslant M(d-c)$. Thus g satisfies a Lipschitz condition and the proof follows immediately from Theorem 2.

REFERENCES

1. R. P. Boas, *A Primer of Real Functions*, 3rd edition, Carus Mathematical Monographs of the MAA, No. 13, 1981, pp. 141–142.
2. M. W. Botsko, An elementary proof that a bounded a.e. continuous function is Riemann integrable, *Amer. Math. Monthly* 95 (1988), 249–252.
3. R. R. Goldberg, *Methods of Real Analysis*, Blaisdell Publishing Co., New York, 1964, pp. 163–164.