that one of them stopped and started the match repeatedly until he found the perfect strategy through trial and error, and the second accessed and analyzed the program to determine the computer's strategy.

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Proving that Three Lines Are Concurrent

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The role of elementary geometry in learning proofs is well established. Among the more challenging problems that a student may encounter, those asking to prove that three lines are concurrent occupy a special place. The common approach in solving these problems is finding a suitable triangle where the three lines are known cevians such as medians or perpendicular bisectors. Yet many other problems are even more general and involve arbitrary concurrent cevians. Ceva's Theorem is a standard approach in this case:

Theorem (Ceva's Theorem). *Given a triangle ABC, and points A', B', and C' that lie on lines BC, CA, and AB respectively, the lines AA', BB', and CC' are concurrent if and only if*

$$\frac{|C'A|}{|C'B|} \cdot \frac{|A'B|}{|A'C|} \cdot \frac{|B'C|}{|B'A|} = 1.$$

Finding a suitable triangle and expressing the ratios in the equation above is not always straightforward. We present an elementary solution to an interesting problem emphasizing two important steps in applying Ceva's Theorem:

Problem. Let ABC be a triangle. Construct rectangles $ACDE$, $AFGB$, and $BHIC$, one on each side of ABC. Prove that the perpendicular bisectors to the segments EF, GH, and ID are concurrent.

This is a particular case of a more general set of problems related to rectangles attached to sides of a triangle [**1**]. The initial difficulty lies in the fact that the problem seems very general. Similar problems usually have additional restrictions, such as squares instead of rectangles or an initial triangle that is not arbitrary. Furthermore, the location of the intersection point of the three lines in the problem seems also arbitrary for different rectangle widths and it does not relate immediately to the triangle ABC.

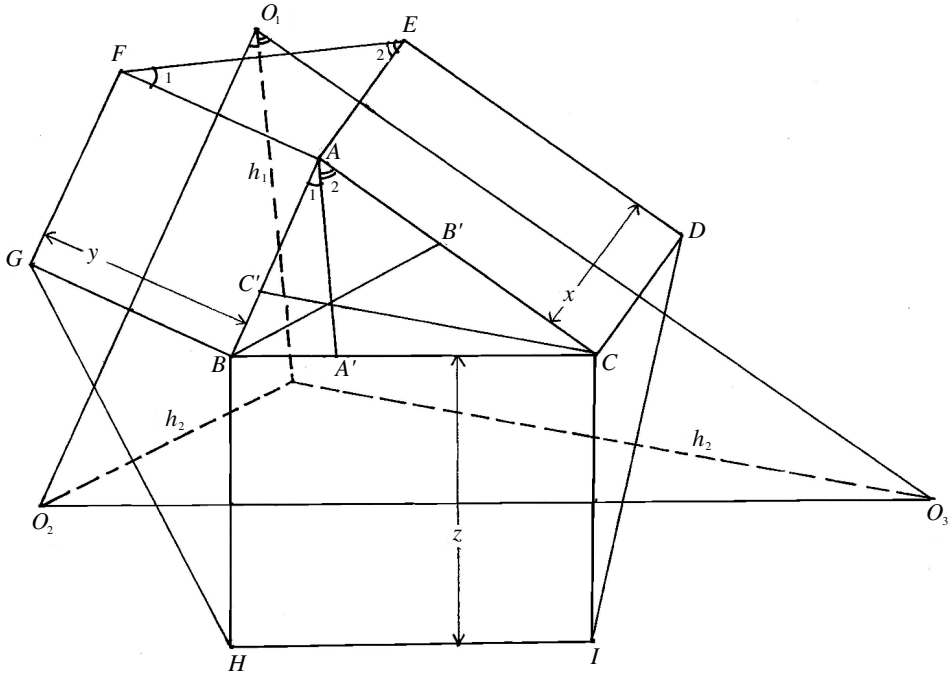

Figure 1.

One overall proof strategy is to attempt to use strictly the given quantities. These are the angles and the lengths of the sides of $\triangle ABC$ and the width of each rectangle, denoted by x, y and z.

Step 1. Identify a triangle where the given lines are cevians.

- We construct the perpendicular bisectors of FA, EA, and CI. These form a triangle $\triangle O_1O_2O_3$. O_1 is the circumcenter of $\triangle EAF$, O_2 is the circumcenter of $\triangle GHB$ and O_3 is the circumcenter of $\triangle CDI$. So the lines given in the problem, h_1, h_2, and h_3, are cevians in $\triangle O_1O_2O_3$ which is similar to $\triangle ABC$.
- We now reduce the original problem to one in $\triangle ABC$ by constructing the following parallels: $AA' \parallel h_1$, $BB' \parallel h_2$, and $CC' \parallel h_3$. All we have to do now is to prove that AA', BB', and CC' are concurrent.

Step 2. Use Ceva's Theorem. To ensure that it will work we need to use only the elements in $\triangle ABC$ and x, y, and z when we express the ratios in Ceva's Theorem.

We will use repeatedly the Law of Sines as follows:
In $\triangle ABA'$ and $\triangle ACA'$:

$$\frac{|A'B|}{|AA'|} = \frac{\sin\angle A_1}{\sin\angle B}, \quad \frac{|A'C|}{|AA'|} = \frac{\sin\angle A_2}{\sin\angle C},$$

which leads to

$$\frac{|A'B|}{|A'C|} = \frac{\sin\angle A_1}{\sin\angle A_2} \cdot \frac{\sin\angle C}{\sin\angle B}.$$

Now notice that

$$\angle A_1 \equiv \angle(O_1O_2, h_1) \equiv \angle F_1,$$
$$\angle A_2 \equiv \angle(O_1O_3, h_1) \equiv \angle E_2.$$

Using the Law of Sines in $\triangle EFA$ we obtain

$$\frac{\sin\angle A_1}{\sin\angle A_2} = \frac{\sin\angle F_1}{\sin\angle E_2} = \frac{x}{y}.$$

Finally,

$$\frac{|A'B|}{|A'C|} = \frac{x}{y} \cdot \frac{\sin\angle C}{\sin\angle B}.$$

Now we are able to write directly the other two ratios in Ceva's Theorem using circular permutations:

$$\frac{|B'C|}{|B'A|} = \frac{y}{z} \cdot \frac{\sin\angle A}{\sin\angle C},$$
$$\frac{|C'A|}{|C'B|} = \frac{z}{x} \cdot \frac{\sin\angle B}{\sin\angle A}.$$

In conclusion,

$$\frac{|A'B|}{|A'C|} \cdot \frac{|B'C|}{|B'A|} \cdot \frac{|C'A|}{|C'B|} = \frac{x}{y} \cdot \frac{y}{z} \cdot \frac{z}{x} \cdot \frac{\sin\angle C}{\sin\angle B} \cdot \frac{\sin\angle A}{\sin\angle C} \cdot \frac{\sin\angle B}{\sin\angle A} = 1.$$

which proves that AA', BB', CC' and, therefore, h_1, h_2, h_3 are concurrent.

As a final remark, once we identify the triangle where the given lines are cevians, the effort should be directed to express each ratio in Ceva's Theorem using the original quantities only. In other words, one should write the three ratios using strictly the given hypotheses and avoiding statements inferred from them. This almost always guarantees the circular permutation and the necessary cancellation in the final part of the proof.

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