What Goes Up Must Come Down; Will Air Resistance Make It Return Sooner, or Later?

JOHN LEKNER
Victoria University
Wellington, New Zealand

A ball thrown straight up with speed v_i would, in the absence of air, return in time $2v_i/g$. Air resistance, or drag, will influence the return time in two ways: the maximum height reached is less than the zero-drag height $v_i^2/2g$, and the speed at any height z is less than the zero-drag speed. (These statements follow from the energy equation $\frac{1}{2}mv_i^2 = \frac{1}{2}mv(z)^2 + mgz + W$, where m is the mass of the ball, and W is the (positive) work done against air resistance. The speed is zero at the top of the trajectory, so $z_{\max} < v_i^2/2g$; and at any z, $v(z) < \sqrt{v_i^2 - 2gz}$. Note that the energy conservation equation is not an additional physical principle: it follows from the equation of motion on multiplying by v and integrating.) Thus with air resistance, the ball has a *shorter distance* to travel, but at a *slower speed*. Which effect wins?

Let $f(v)$ be the deceleration due to the drag force. The equation of motion then reads $dv/dt = -g - f(v)$ on the way up, and $dv/dt = g - f(v)$ on the way down (it is convenient to deal with speeds rather than velocities in this context). We will assume that $f(v)$ has the property that there is just one speed at which the gravitational and drag forces are in balance. This defines the *terminal speed* v_t: $f(v_t) = g$. The terminal speed is a natural scaling parameter for this problem. Let $u = v/v_t$ and $\phi(u) = f(v)/f(v_t) = f(v)/g$. Then by integrating dt (obtained from the equation of motion) we find the time to go up to maximum height is

$$t_{\text{up}} = \int_0^{v_i} \frac{dv}{g + f(v)} = \frac{v_t}{g} \int_0^{u_i} \frac{du}{1 + \phi(u)}, \tag{1}$$

and the time to come down is

$$t_{\text{down}} = \int_0^{v_f} \frac{dv}{g - f(v)} = \frac{v_t}{g} \int_0^{u_f} \frac{du}{1 - \phi(u)}. \tag{2}$$

The speed on impact, v_f, is determined by the condition that the distance travelled on the way up is the same as that travelled on the way down. These are given by integrating $v\,dt$; we find u_f is determined by

$$\int_0^{u_i} \frac{u\,du}{1 + \phi(u)} = \int_0^{u_f} \frac{u\,du}{1 - \phi(u)}. \tag{3}$$

We are interested in the ratio τ of the return time to the zero-drag return time $2v_i/g$. From (1) and (2),

$$\tau = \frac{t_{\text{up}} + t_{\text{down}}}{2v_i/g} = \frac{1}{2u_i}\left[\int_0^{u_i} \frac{du}{1+\phi} + \int_0^{u_f} \frac{du}{1-\phi}\right]. \tag{4}$$

Physically, $f(v)$ must go to zero as v goes to zero. Thus Φ, the maximum value of $\phi(u)$, can be made arbitrarily small compared to unity when the initial speed v_i is chosen sufficiently small compared to the terminal speed v_t (u_i sufficiently small). We can therefore expand $[1 \pm \phi(u)]^{-1}$ in (3) and (4), to find

$$\frac{u_f}{u_i} = 1 - \frac{2}{u_i^2} \int_0^{u_i} u\phi\,du + O(\Phi^2)$$

$$\tau = 1 - \frac{1}{u_i^2} \int_0^{u_i} u\phi\,du + O(\Phi^2). \tag{5}$$

Thus any physically reasonable form of drag will make the ball return sooner, provided the launch speed is small compared to the terminal speed.

Wind tunnel experiments [**1**] on spheres show that the drag force is (approximately) proportional to v^2 in the Reynolds number range $10^3 \leqslant R \leqslant 10^5$. This covers the range of practical interest, provided the launch speeds are kept moderate (a sphere of diameter 1.5 cm and speed 10^3 cm/s has $R \cong 10^4$ in air). For $f = kv^2$ ($\phi = u^2$) we find from (3) and (4) that

$$u_f = \frac{u_i}{\sqrt{1+u_i^2}} \tag{6}$$

and

$$\tau = \left(\arctan u_i + \operatorname{arctanh} u_f\right)/2u_i. \tag{7}$$

The numerator $N(u) = \arctan u + \operatorname{arctanh}\left(u/\sqrt{1+u^2}\right)$ has slope $dN/du = (1+u^2)^{-1} + (1+u^2)^{-1/2}$, which is less than 2 for nonzero u. Thus $N(u_i)$ increases more slowly than $2u_i$, the leading term in its Taylor expansion about $u_i = 0$. It follows that, for a v^2 drag, τ is always less than unity, no matter what the initial speed.

Could this result be true for an arbitrary (nonnegative) drag $f(v)$? Let's try a few more examples. When f is linear in v (Stokes' law), we find the attractive result

$$\tau = \tfrac{1}{2}\left(1 + \frac{u_f}{u_i}\right)$$

or

$$t_{\text{up}} + t_{\text{down}} = \frac{v_i + v_f}{g}. \tag{8}$$

Since v_f is always less than v_i, we again have the return time being shortened by air resistance, irrespective of the initial speed.

So far, all has indicated a shorter return time. Now consider some fractional powers. First suppose $f(v) \sim v^{1/2}$. Setting $u = w^2$, we find that u_f is determined by an interesting transcendental equation

$$\frac{1}{3}w_i^3 - \frac{1}{2}w_i^2 + w_i - \log(1+w_i) = -\frac{1}{3}w_f^3 - \frac{1}{2}w_f^2 - w_f - \log(1-w_f), \tag{9}$$

and that the ratio of return time to zero-drag return time is

$$\tau = w_i^{-2}\{w_i - \log(1+w_i) - w_f - \log(1-w_f)\}. \tag{10}$$

For $u_i \gg 1$ we find $\tau \to \frac{1}{3}u_i^{1/2}$, larger than unity.

Next, suppose $f(v) \sim v^{2/3}$. Setting $u = y^3$ we find

$$\frac{1}{2}y_i^4 - y_i^2 + \log(1+y_i^2) = -\frac{1}{2}y_f^4 - y_f^2 - \log(1-y_f^2) \tag{11}$$

and

$$\tau = \frac{3}{2y_i^3}\{y_i - \arctan y_i + \operatorname{arctanh} y_f - y_f\}. \tag{12}$$

For $u_i \gg 1$, $\tau \to \frac{3}{8}u_i^{1/3}$, again larger than unity.

The above results suggest to me that there is a cross-over at the linear force law:

CONJECTURE. *For powers p in $f(v) = kv^p$, $p \geqslant 1$ gives a return time which is always shorter than the zero-drag return time $2v_i/g$. For $p < 1$, the return time is shorter for small initial speeds, but eventually becomes longer than $2v_i/g$ as v_i increases. The closer p is to 1, the higher the ratio of the initial speed to the terminal speed before this happens.*

We have determined $\tau(u_i)$ for only four values of p: 2, 1, 1/2, 2/3. Students may enjoy some of the following projects in analysis and numerical methods:

(a) plotting τ versus u_i for these four values of p;

(b) finding other values of p for which the integral equation (3) for u_f is reducible to a transcendental equation, and plotting $\tau(u_i)$ for these;

(c) a class exercise in which different values of $p < 1$ are assigned to students or student groups, and each is asked to find the u_i for which $\tau = 1$.

This problem originated from a first-year physics question set by Tim Shirtcliffe. I am grateful to him and to John Harper and Graeme Wake for helpful comments.

Reference

[1] G. K. Batchelor, An Introduction to Fluid Dynamics, Cambridge, 1967, p. 341.

A Method of Duplicating the Cube

EDWARD V. GRAEF
1814 Kent Road
Pittsburgh, PA 15241

V. C. HARRIS
San Diego State University
San Diego, CA 92182

The old problem of duplicating the cube—that is, of constructing a cube with volume twice that of a given cube—was solved geometrically in several ways by the ancient Greek mathematicians (see Eves [**1**] for a summary). It is the purpose of this note to show how analytic geometry can be used to construct two curves which will give one more solution to the problem.

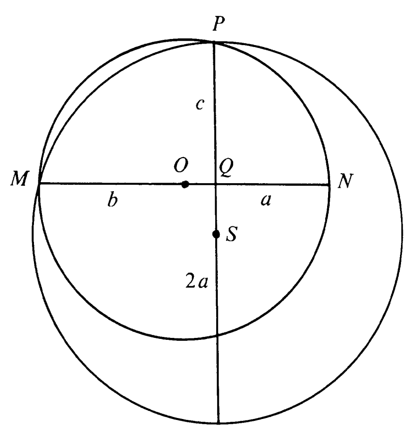

FIGURE 1