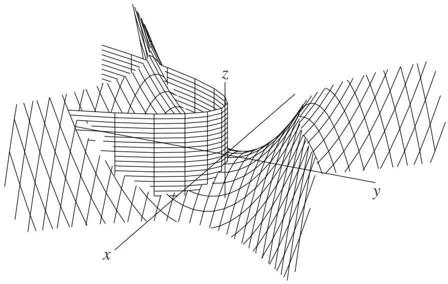

Figure 5.

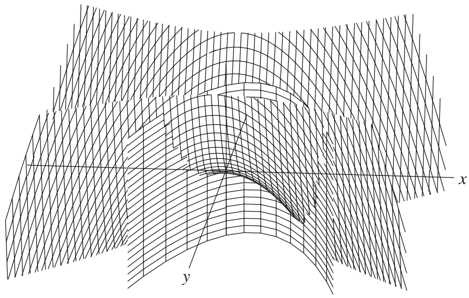

Figure 6.

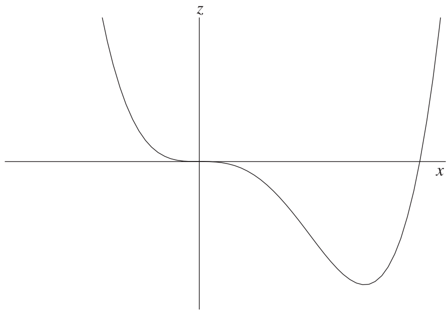

Figure 7.

Conic Sections from the Plane Point of View

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We give an analytic proof of the fact that the conic sections are obtained by cutting a cone at various angles. Our proof does not involve spheres or circles (see [**1, 3, 4**]), but primarily depends upon the cutting plane itself.

Figure 1 shows a two-napped circular cone C which may be viewed as the result of rotating the line g (generator) about the fixed line l (z-axis) while maintaining the same angle (β) between g and l. We choose the intersection o of g and l as the origin. Let $P(x, y, z)$ be a point on the surface of C. From Figure 1 (see also [**2**, p. 751]), we see that $\frac{a}{r} = \frac{c}{z}$. It follows that the equation of C is

$$x^2 + y^2 = z^2 \tan^2 \beta. \tag{1}$$

Suppose a cutting plane E that does not contain o makes an angle α with the z-axis. The angle between E and the xy-plane is $\frac{\pi}{2} - \alpha$. Let the equation of plane E be

$$z = \tan\left(\frac{\pi}{2} - \alpha\right) y + h \qquad (h \neq 0). \tag{2}$$

Substituting (2) into (1) gives the equation of the intersection of C and E:

$$x^2 + y^2 - \tan^2 \beta \left[\tan\left(\frac{\pi}{2} - \alpha\right) y + h\right]^2 = 0,$$

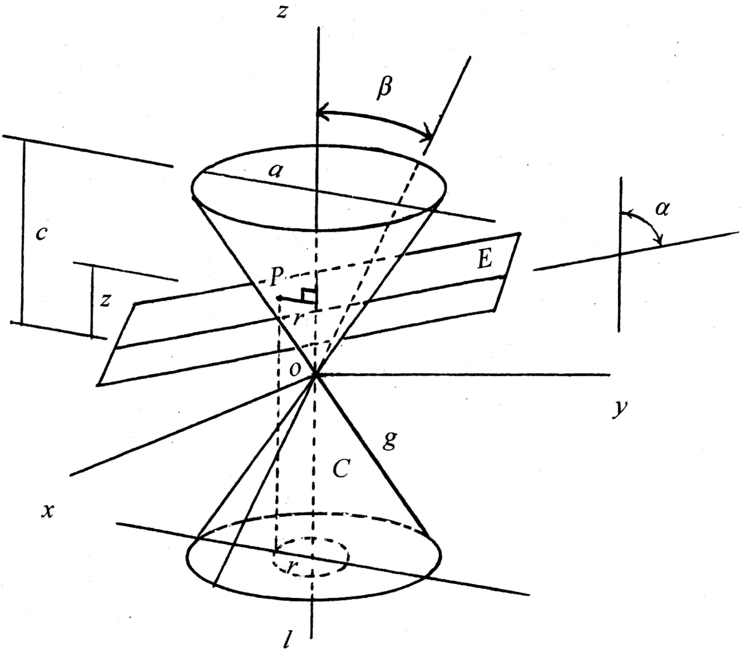

Figure 1.

or

$$x^2 + \left[1 - \tan^2\beta\tan^2\left(\frac{\pi}{2} - \alpha\right)\right] y^2 - 2h\tan^2\beta\tan\left(\frac{\pi}{2} - \alpha\right) y = h^2\tan^2\beta. \quad (3)$$

Thus,

(i) If $\beta < \alpha \le \frac{\pi}{2}$, then $0 < \frac{\pi}{2} - \alpha < \frac{\pi}{2} - \beta$. So $1 - \tan^2\beta\tan^2(\frac{\pi}{2} - \alpha) > 0$. The coefficients of x^2 and y^2 are positive. Hence the conic is an ellipse. (If $\alpha = \frac{\pi}{2}$, this is a circle.)

(ii) If $\alpha = \beta$, then the coefficient of y^2 is zero. Equation (3) reduces to $x^2 - (2h\tan\beta)y = h^2\tan^2\beta$, and we have a parabola.

(iii) If $0 \le \alpha < \beta$, then $0 < \frac{\pi}{2} - \beta < \frac{\pi}{2} - \alpha$. Thus, $1 - \tan^2\beta\tan(\frac{\pi}{2} - \alpha) < 0$. The coefficients of x^2 and y^2 have opposite signs. The conic is a hyperbola.

It should be noted that if plane E contains o, the intersection of E and the cone is a point, a line, or a pair of intersecting lines corresponding to $\alpha = \frac{\pi}{2}$ or $\alpha > \beta$, $\alpha = \beta$, and $0 \le \alpha < \beta$, respectively (see [**2**, p. 637]). These are called degenerate conics.

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