A New Proof of the Double Butterfly Theorem

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The butterfly problem, which dates back to at least 1815, seems to hold a continual interest. It has had a variety of proofs and extensions in recent years. A synopsis of this celebrated problem is given in [4] and a very comprehensive update is given in [1]. In this note, we provide a new proof of one of its extensions—namely, the double butterfly theorem.

This theorem, as given by D. Jones in [3], is

Double Butterfly Theorem. *Let PQ be a fixed chord of a circle and let "butterfly R" and "butterfly S" be inscribed in the circle and oriented such that their wings cut PQ (in order from left to right) at* R_4, R_3, R_2, R_1 *and* S_1, S_2, S_3, S_4 *respectively. If* $PR_1 = QS_1$, $PR_2 = QS_2$, *and* $PR_3 = QS_3$, *then* $PR_4 = QS_4$ *(see* Figure 1).

For our proof we use a lemma attributed to Hiroshi Haruki [2].

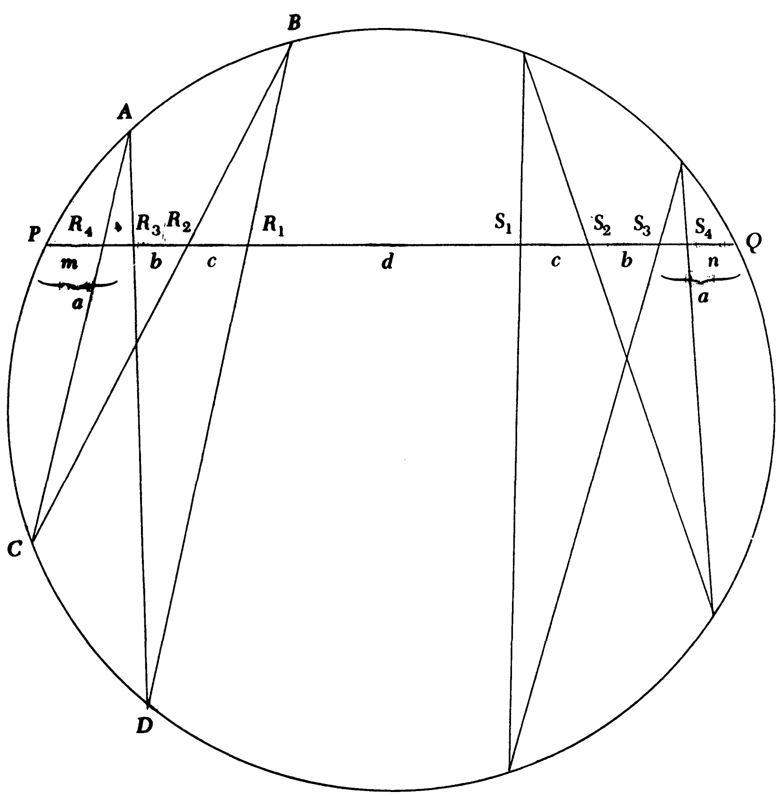

FIGURE 1

LEMMA. *Suppose PQ and CD are non-intersecting chords in a circle and that B is a variable point on the arc PQ remote from C and D. Then for each position of B, the lines BC and BD cut PQ into segments of lengths x, y, z where xz/y is a constant (see* FIGURE 2).

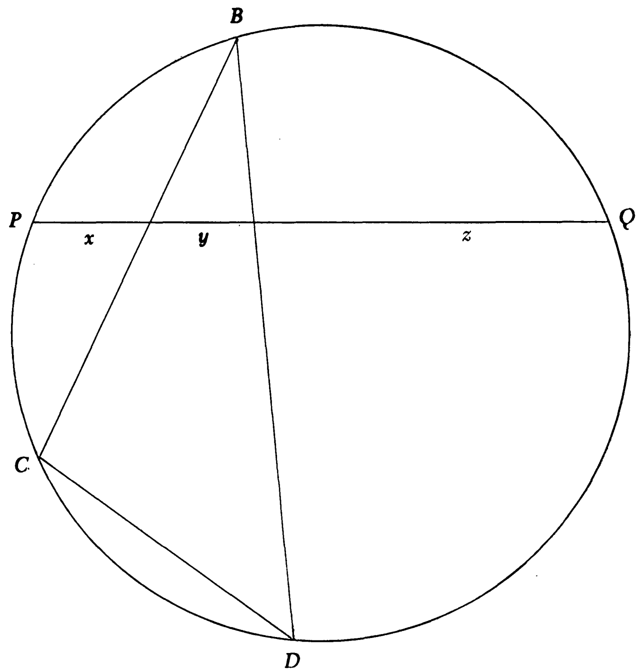

FIGURE 2

This lemma, which deserves a much wider recognition, is proved in [2] by elementary geometry and in [1] by cross ratios. For our proof of the theorem we let $m = PR_4$, $n = QS_4$, $a = PR_3 = QS_3$, $b = R_3R_2 = S_3S_2$, $c = R_2R_1 = S_2S_1$, $d = R_1S_1$, and $e = a + b + c + d$ as seen in FIGURE 1. By applying Haruki's lemma twice to points A and B and fixed chords PQ and CD of butterfly R, we have

$$\frac{m(b+c+e)}{a-m} = \frac{(a+b)e}{c},$$

since both are equal to the same constant. Similarly

$$\frac{n(b+c+e)}{a-n} = \frac{(a+b)e}{c}$$

for butterfly S. By solving these two equations simultaneously, we obtain $m = n$ which completes the proof.

REFERENCES

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3. Dixon Jones, A double butterfly theorem, this MAGAZINE 49 (1976), 86–87.
4. Leo Sauvé, The celebrated butterfly problem, *EUREKA* (now *CRUX MATHEMATICORUM*) (1976), 2–5.