The optimal angle θ_c depends on the velocity ratio r and is independent of the street width w. For walkers without calculators, Figure 3 plots θ_c as a function of r.

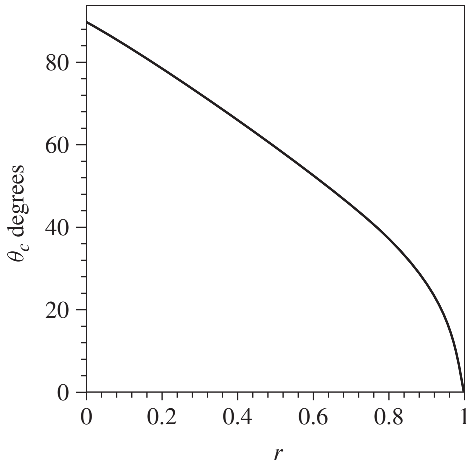

Figure 3. Optimal initial angle versus speed ratio.

For the parameters $w = 30$ ft., $v_f = 5$ ft./sec., $v_s = 3$ ft./sec., as in our example, the optimal solution corresponds to $\theta = \arccos(3/5) \cong 53°$. The distance along AC between meetings will be 45 ft. and the time between meetings will be 15 sec.

Summary. Fay and Sam go for a walk. Sam walks along the left side of the street while Fay, who walks faster, starts with Sam but walks to a point on the right side of the street and then returns to meet Sam to complete one segment of their journey. We determine Fay's optimal path minimizing segment length, and thus maximizing the number of times they meet during the walk. Two solutions are given: one uses derivatives; the other uses only continuity.

Acknowledgment. Our thanks to the reviewers and editors for several very helpful suggestions.

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The Cobb-Douglas Function and Hölder's Inequality

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Whenever I teach Business Calculus, I am struck that the Cobb-Douglas production function is ripe for an application of Hölder's inequality [**2**]. This capsule explores an example.

http://dx.doi.org/10.4169/college.math.j.42.5.387
MSC: 26D15, 91B38

Cobb-Douglas The Cobb-Douglas function Q expresses the rate of production (of an economic entity) in terms of inputs of capital K and labor L. It has the form $Q = AK^{\alpha}L^{1-\alpha}$, where A is a scalar, $0 < \alpha < 1$, and Q, K, and L are nonnegative [**1**]. The constant A is total factor productivity and accounts for effects in the output not caused by capital and labor, such as technology, efficiency, or, for that matter, good weather. Economic considerations dictate the form of the function, as is explained succinctly in [**3**, pp. 887–888]. The exponents of K and L sum to one in this model, which means we are assuming "constant returns to scale"; that is, for example, if K and L are each increased 15%, so is output. The exponents also give capital's and labor's respective shares of output. If $\alpha = 1/5$, capital has a 20% share of output and labor an 80% share. These are *not* indicative of their share of a budget; rather they point to respective contribution to the output, per unit, per unit time. The model assumes α is constant, which is perfectly plausible for relatively short time intervals.

Hölder Imagine that the inputs are adjusted over time, so that K and L depend on time t. To find the total output over a time interval, say $[0, T]$, we would calculate $\int_0^T Q(t)\,dt$, perhaps with some difficulty. An upper bound on production can be found easily by applying Hölder's inequality:

$$\int_a^b |f(x)g(x)|\,dx \le \left(\int_a^b |f(x)|^p\,dx\right)^{1/p} \left(\int_a^b |g(x)|^q\,dx\right)^{1/q},$$

where p, q are real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Setting $p = \frac{1}{\alpha}$ and $q = \frac{1}{1-\alpha}$ we get

$$\int_0^T Q(t)\,dt = \int_0^T A\,(K(t))^{\alpha}\,(L(t))^{1-\alpha}\,dt \le A\left(\int_0^T K(t)\,dt\right)^{\alpha} \left(\int_0^T L(t)\,dt\right)^{1-\alpha}.$$

Example Suppose a firm has weekly production $Q = 1125K^{1/5}L^{4/5}$, implying total factor productivity A of 1125, and a capital-labor cost ratio of one to four. Suppose, in addition, labor unit costs of \$180/week, a capital rental rate K of \$225/week, and a budget of \$90,000/week. Then maximizing production leads to an allotment of 80 units of capital and 400 units of labor. The firm anticipates an increasing cash flow sufficient to budget an additional \$2070 each week for the next 4 weeks, which means it can allocate more to each of K and L, but not necessarily in the 1-to-5 proportion (in terms of units) that initially maximized production. Assuming the labor market can supply no more than an additional 9 units per week, the best the firm can do is increase capital by 2 units and labor by 9 units per week. Accordingly, $K(t) = 80 + 2t$ and $L(t) = 400 + 9t$, as functions of time. Total production over that time would be

$$\begin{aligned}\int_0^4 Q(t)\,dt &= \int_0^4 1125(80 + 2t)^{1/5}(400 + 9t)^{4/5}\,dt \\ &\le 1125\left(\int_0^4 (80 + 2t)\,dt\right)^{1/5} \left(\int_0^4 (400 + 9t)\,dt\right)^{4/5} \\ &= 1125(336)^{1/5}(1672)^{4/5} \\ &\approx 1{,}364{,}613.\end{aligned}$$

Using this procedure a production manager can provide upper management with a useful (over) estimate of production for the next 4 weeks with a simple hand-held calculator. Such “back of the envelope” calculations can direct appropriate budgeting for a desired production output.

Afterword Our estimate turns out to be only a dollar off. Why is the upper bound so close to the actual answer? Equality holds in Hölder’s inequality if

$$\alpha|f|^p = \beta|g|^q$$

for some α and β not both 0. In our example, while this condition does not hold, the two functions are not dramatically far from satisfying it.

We invite the reader to solve the following problem:

Exercise. Short term available capital and labor are anticipated to obey the following equations: $K(t) = 500e^{0.5t}$ and $L(t) = 60t + 1$. These formulas represent quick growth in a company that expects exponential investment in capital and large linear growth in labor costs each week, modeled continuously. Time is in weeks and production units may be thought of as hundreds of dollars. The company believes its production function to be $Q = 200K^{1/3}L^{2/3}$. Use Hölder’s inequality to find an upper bound for the company’s anticipated production for the coming four weeks. This approach gives an answer with an error of roughly $\frac{1}{2}\%$.

Summary. Hölder’s inequality is here applied to the Cobb-Douglas production function to provide simple estimates to total production.

Acknowledgment. The author would like to thank the referee and the editors for their thoughtful suggestions; they improved this capsule decidedly.

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The Center of Mass of a Soft Spring

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Finding the center of mass is a fundamental part of the mechanics of rigid bodies. The center of mass of highly symmetric rigid bodies is easily obtained. However, if the shape and size of a body is affected by external factors, like gravity, its center of mass changes in a way that it is sometimes difficult to predict. This is the case with a soft helical spring. When such a spring is hung vertically it stretches non-uniformly, and its center of mass is no longer located at the center of the spring’s length.

In this article, we find the center of mass of a vertically suspended soft spring (a Slinky) using the calculus. Here, we use the term “soft” to describe a spring whose stiffness is small enough so that its own weight stretches it noticeably when it is suspended vertically. Our results are in excellent agreement with experimental results and non-analytic approaches. The method we use is easily incorporated into any undergraduate calculus or general physics course.

http://dx.doi.org/10.4169/college.math.j.42.5.389
MSC: 00A69, 00A79, 74A99