x, y. We refer to the coefficients in the Maclaurin series as the loss sequence. The table below gives some sample results. Taking $(x, y) = (5, 1)$, the table shows that L_5, L_7, and L_9 have 1, 4, and 13 elements, respectively. If we take $(x, y) = (4, 2)$, the table shows that L_4, L_6, and L_8 have 1, 4, and 13 elements, respectively.

x	y	Loss Function $g(u)$	Loss Sequence—first ten terms
1	2	$1/(1-u)$	1, 1, 1, 1, 1, 1, 1, 1, 1, 1
1	3	$(1-u)/(1-2u)$	$1, 1, 2, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8$
1	4	$(1-2u)/(1-3u+u^2)$	1, 1, 2, 5, 13, 34, 89, 233, 610, 1597
5	1	$1/(1-4u+3u^2)$	1, 4, 13, 40, 121, 364, 1093, 3280, 9841, 29524
4	2	$1/(1-4u+3u^2)$	1, 4, 13, 40, 121, 364, 1093, 3280, 9841, 29524
3	3	$1/(1-3u)$	$1, 3, 3^2, 3^3, 3^4, 3^5, 3^6, 3^7, 3^8, 3^9$

We have been able to show that the radii of convergence r of the Maclaurin series of all loss functions obey $1/4 \leq r < 1$. The case $(x, y) = (1, 1)$ is an exceptional case, since in this case, we do not get an infinite sequence since the game then ends in one flip. For $n > 1$, the pairs $(x, y) = (n, 1)$ and $(x, y) = (n-1, 2)$ yield the same loss function. Also, for any particular loss function, there can be only finitely many pairs (x, y) that have this same loss function. The loss sequence for $(x, y) = (1, 4)$ yields the odd indexed Fibonacci numbers (plug Fibonacci numbers or Fibonacci polynomials into an internet search).

Acknowledgements. The authors would like to thank the referees, the editor, and Emeric Deutsch for valuable suggestions.

REFERENCES

1. W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. 1, 3rd ed., John Wiley, 1968.
2. J. Bak, The Recreational Gambler: Paying the Price for More Time at the Table, this MAGAZINE **80** (2007) 183–194.
3. J. Harper and K. Ross, Stopping Strategies and Gambler's Ruin, this MAGAZINE **78** (2005) 255–269.
4. E. Parzen, *Modern Probability Theory and Its Applications*, Wiley Classics, 1992.
5. R. I. Jewett and K. A. Ross, Random Walks on Z, *College Math. J.* **19** (1988) 330–342.
6. S. Ross, *A First Course in Probability*, 5th Edition, Prentice Hall, 1997.

Lazy Student Integrals

GREGORY GALPERIN
GREGORY RONSSE
Eastern Illinois University
Charleston, IL 61920
ggalperin@eiu.edu
gronsse@eiu.edu

A challenging integral

Let $\alpha \in \mathbb{R}$ and consider the problem of evaluating

$$I(\alpha) = \int_0^\infty \frac{dx}{(1+x^\alpha)(1+x^2)}$$

as a function of α. As a first attempt, we might substitute $x = \tan\theta$ to obtain the integral

$$\int_0^{\pi/2} \frac{\cos^\alpha\theta \, d\theta}{\cos^\alpha\theta + \sin^\alpha\theta}. \tag{1}$$

This seems no better than the original. As a second attempt, split the integral over intervals [0, 1] and [1, ∞). On the second interval substitute $u = 1/x$ to obtain

$$I(\alpha) = \int_0^1 \frac{dx}{(1+x^\alpha)(1+x^2)} + \int_0^1 \frac{u^\alpha du}{(1+u^\alpha)(1+u^2)}.$$

Replacing the dummy variable of integration u with x and combining the two integrals, we obtain

$$I(\alpha) = \int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4},$$

which is independent of α.

Many people, at first glance, think we should obtain a decreasing function of α and find it surprising that $I(\alpha)$ is constant. It is a trick of the mind. Upon looking at an integral of a bounded function over $[0, \infty)$, we tend to think of the behavior of the function for large values of x and ignore the behavior for $0 \leq x \leq 1$. The natural symmetry of inversion between [0, 1] and [1, ∞) reveals that the integral over each of these two intervals must sum to a constant.

Now, of course, the integral in equation (1) is the constant $\pi/4$, independent of α. Is there symmetry here?

Tale of the lazy student

In first semester calculus class, students learn to evaluate definite integrals by finding anti derivatives. In order to remind them that an integral is a limit of Riemann sums, it is wise for an instructor to ask them to evaluate an integral similar to

$$\int_{-3}^{3} \frac{\sin x^3 dx}{\sqrt{1+x^4} + \cos(2x)}.$$

The answer is, of course, 0. We are integrating an odd function over an interval which is symmetric about 0. The area above the x-axis is equal to the area below the x-axis.

The lazy student, upon seeing such complicated integrals, has become conditioned to write down 0 immediately and get the right answer. He has noticed that such problems always seem to have positive and negative portions that cancel each other. The instructor must grudgingly admire this valid insight, but he seeks to enforce more careful analysis by altering the problem. So he adds a constant to the integrand, but keeps it mysterious by combining the constant with the fraction to keep the previous denominator but alter the numerator. Our lazy, but perceptive, student now notices a new rule. "Complicated integrals" can be evaluated by evaluating the integrand at 0 and then multiplying by the length of the interval. The exasperated instructor throws in another gimmick by translating the integral along the x-axis by translation to an interval $[a, b]$ in order to disguise the symmetry. The "good students" are completely baffled and angry. However, our lazy, but ingenious, student rises to the occasion. He evaluates the integrand at the interval's midpoint and multiplying by the length of the interval.

The lazy student's method will not, of course, work for all "complicated integrals". Nevertheless, the lazy student would consider the value of the integral in equation (1) as obvious (provided this student was not too lazy in algebra and trig class). Another formula that is obvious to the lazy student is

$$\int_0^{10} \frac{(3 + x^{\sqrt{7}})\,dx}{6 + x^{\sqrt{7}} + (10 - x)^{\sqrt{7}}} = 5. \tag{2}$$

Let us endeavor to make these formulas obvious to the non lazy, by generalization and abstraction.

Lazy student formulas

Let $f : [0, a] \longrightarrow \mathbb{R}$ be any continuous function. Substitute $u = a - x$ to obtain

$$\int_0^a f(x)\,dx = \int_0^a f(a - u)\,du.$$

Geometrically, the substitution simply reflects the graph of f about the line $x = a/2$, which obviously leaves the area under the curve invariant.

Now suppose that f satisfies the following **symmetry condition**,

$$f(x) + f(a - x) = 1.$$

Integrating, we obtain the lazy student formula

$$\int_0^a f(x)dx = \frac{a}{2}.$$

This is fine, but how do we obtain functions f that satisfy this restrictive symmetry condition? Set

$$f(x) = \frac{g(x)}{g(x) + g(a - x)},$$

where g is any continuous function on $[0, a]$ such that the denominator of the above does not vanish. It is easily verified that f satisfies the symmetry condition. Furthermore, any function f satisfying the symmetry condition has this form, just take $g = f$.

Equation (2) is now obvious by taking $g(x) = 3 + x^{\sqrt{7}}$ and $a = 10$.

As a special case, note that $\sin(x) = \cos(\pi/2 - x)$, to obtain the lazy student formula:

$$\int_0^{\pi/2} \frac{f(\cos x)dx}{f(\sin x) + f(\cos x)} = \frac{\pi}{4},$$

where f is **any** continuous function defined on [0, 1], such that the denominator in the above integrand does not vanish. This includes the integral in (1) as a special case.

Reference

James Stewart, *Calculus*, 5th ed., Brooks/Cole-Thomson Learning, Belmont, CA, 2003.