CLASSROOM CAPSULES

Doublecakes: An Archimedean Ratio Extended

Vera L. X. Figueiredo (vera@ime.unicamp.br), Margarida P. Mello (margarid@ime.unicamp.br), and Sandra A. Santos (sandra@ime.unicamp.br), IMECC, State University of Campinas, C.P. 6065, Campinas, SP, 13083-970, Brazil

The acclaimed 2:3 ratio between the volumes (and the surface areas) of a sphere and its circumscribing cylinder, established by Archimedes (ca. 287–212 B.C.) in his work *On the Sphere and the Cylinder*, produced the first exact expressions for the volume and surface area of a sphere. It is no wonder that he considered it one of his greatest achievements and instructed his friends to engrave his tombstone with a picture of a sphere inscribed in a cylinder, as depicted in Figure 1, and the 2:3 ratio.

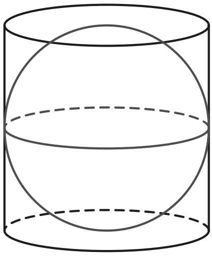


Figure 1. A sphere and its circumscribing cylinder.

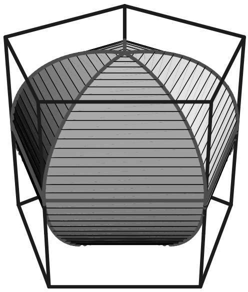


Figure 2. A doublecake and its circumscribing prism.

We extend this result to a new whole family of objects, which we call *doublecakes*, and their circumscribing prisms; an example appears in Figure 2. De Temple [**1**] showed that this ratio holds for a particular pair of this family, the so-called bicylinder (the intersection of two right circular cylinders with orthogonal axes of symmetry) and its circumscribing cube. Bicylinders are featured in most calculus textbooks, and the computation of their volume is a popular exercise. The sphere-cylinder pair is obtained as the limit of doublecake-prism pairs, when the number of sides goes to infinity. It is remarkable, however, that the ratio is valid for all pairs, regardless of the number of sides.

Consider first the object formed by the intersection of n right circular solid cylinders of radius r, whose axes of symmetry are distributed at equal angles around a point on a common plane. Assume the point is the origin O of a three-dimensional coordinate system and the common plane is the xy-plane. The intersection of this object with the xy-plane is thus a regular $2n$-gon with apothem of length r (which may be partitioned into $2n$ equal isosceles triangles sharing the common (central) vertex O). Since each cylinder is symmetric with respect to the xy-plane, so is the resulting intersection. Consider the top half of it, that is, the portion of the intersection in the upper half space. We ask the reader to interpret this half of the object as a *cake* which can be divided into $2n$ equal slices. The base of each slice is thus an isosceles triangle. Figure 3 depicts the intersection of four cylinders (which we henceforth call an 8-sided *doublecake*), its two halves, and one slice from the 8-sided cake.

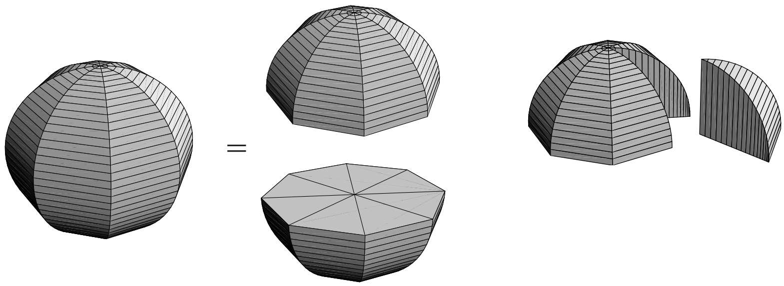


Figure 3. The 4-cylinder intersection (doublecake), 8-sided cake, and a slice.

Using the cake idea we can easily build similar shapes with an odd number of sides, which of course do not correspond to cylinder intersections. One simply cuts a slice of the appropriate size and shape from a cylinder, and builds the cake with copies thereof. Thus this method of construction is more general since it also encompasses the objects resulting from cylinder intersections. The construction of 3- and 5-sided cakes is illustrated in Figure 4. Only half of the exterior surface of a cylinder is shown, so that we can see the wire frame of the slice's boundary. Notice that the plane of symmetry of the slice is orthogonal to the axis of symmetry of the cylinder. The projection of the slice on the horizontal plane is an isosceles triangle with central angle $\widehat{O} = 2\pi/n$.

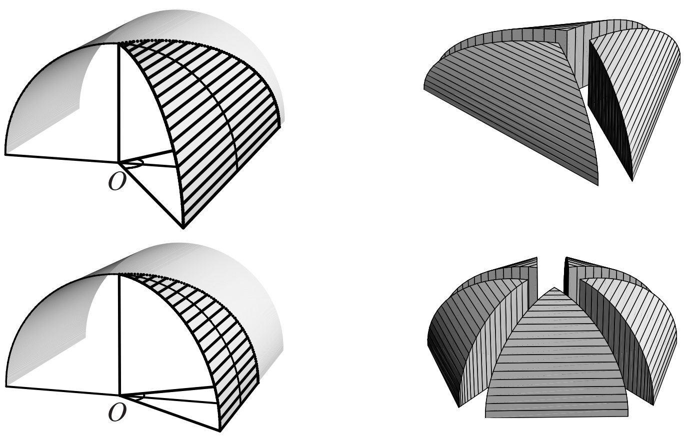


Figure 4. Construction of 3- and 5-sided cakes.

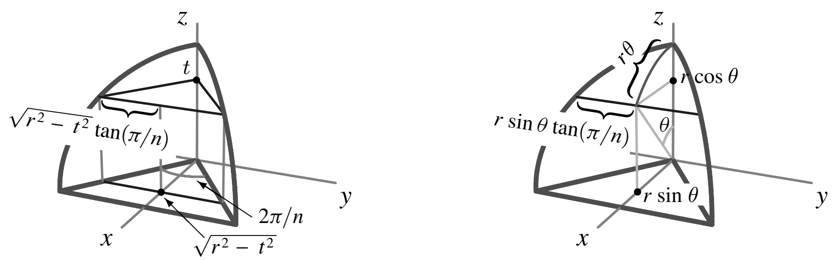
The height of this triangle is the apothem of the regular n-gon that forms the basis of the cake. The *n-sided doublecake* is the union of the n-sided cake and its mirror image through the xy-plane.

De Temple [**1**] established Archimedes' result for the bicylinder intersection (what we call a 2-sided doublecake) and its circumscribing cube. The natural analogue for the n-sided doublecake is the circumscribing prism. This is also n-sided, and has the same n-gon as the inscribed doublecake for its cross-section. The top and bottom vertices of the doublecake should belong to the bases of the circumscribing prism, whose cross-section is the same n-gon as in the doublecake base. Figure 2 shows the 5-sided doublecake and its circumscribing prism.

From the construction we conclude that the volume of an n-sided doublecake is $2n$ times that of a slice and the surface area of the n-sided doublecake is $2n$ times that of the curved face of a slice. If we cut the slice with horizontal planes, every cross-section of the slice is an isosceles triangle similar to the one at the base. Thus, for $z = t$, the height of the triangle is $\sqrt{r^2 - t^2}$ and the base of the triangle measures $2\sqrt{r^2 - t^2}\,\tan(\pi/n)$. Therefore the volume (Figure 5(a)) of the n-sided doublecake is

$$V_{dc} = 2n \int_0^r (r^2 - z^2)\tan\left(\frac{\pi}{n}\right) dz = \frac{4}{3} n r^3 \tan\left(\frac{\pi}{n}\right). \tag{1}$$

Formula (1) extends the result obtained on Lo Bello [**2**], in which only the case of n even was considered.



(a) (b)

Figure 5. Calculating the volume (a) and surface area (b) of a slice.

In order to calculate the area, we recall that the cylinder surface can be constructed by rolling a flat surface. The surface of the slice can be calculated by unfolding and obtaining the corresponding plane region. Suppose we unfold the slice surface so that the arc of the radius-r circumference along the middle of the slice coincides with the line $y = 0$, $z = r$, parallel to the x-axis; see Figure 5(b). Then when the length along this arc is $\ell = r\theta$, the x and z coordinates are $x = r\sin\theta$ and $z = r\cos\theta$, respectively. Since the slice comes from a flat surface, the intersection thereof with its tangent plane at the point $(x, y, z) = (r\sin\theta, 0, r\cos\theta)$ is a segment parallel to the y-axis, with length $2r\sin\theta\tan(\pi/n)$. Thus the unfolded slice is the region between the curves $y = \pm r\sin(\theta)\tan(\pi/n) = \pm r\sin(\ell/r)\tan(\pi/n)$ for $0 \le \ell \le r\pi/2$. The area of the n-sided doublecake is thus

$$S_{dc} = 2n \int_0^{r\pi/2} 2r\sin\left(\frac{x}{r}\right)\tan\left(\frac{\pi}{n}\right) dx = 4nr^2\tan\left(\frac{\pi}{n}\right). \tag{2}$$

The geometric reasoning behind the integrals in (1) and (2) is depicted in Figure 5.

The n-sided double cake is circumscribed by an n-sided right prism, whose basis is the regular n-gon with apothem r, as shown in Figure 2. Thus the volume and surface area of the circumscribing prism are given by

$$V_p = n\text{-gon area} \cdot \text{prism height} = nr^2 \tan\left(\frac{\pi}{n}\right) \cdot 2r = 2nr^3 \tan\left(\frac{\pi}{n}\right) \tag{3}$$

and

$$\begin{aligned} S_p &= 2n\text{-gon area} + n\text{-gon perimeter} \cdot \text{prism height} \\ &= 2nr^2 \tan\left(\frac{\pi}{n}\right) + 2nr \tan\left(\frac{\pi}{n}\right) \cdot 2r \\ &= 6nr^2 \tan\left(\frac{\pi}{n}\right). \end{aligned} \tag{4}$$

Formulas (1)–(4) imply that the ratio $2{:}3 = V_{dc}{:}V_p = S_{dc}{:}S_p$ between volumes and surface areas is valid for these objects for all $n \geq 3$.

Moreover, as $n \to \infty$, the n-sided doublecake tends to a sphere of radius r, while the corresponding prism tends to its circumscribing cylinder. These limit processes furnish alternative methods for computing the volume and surface area of a sphere. Both limits involve the indeterminate form $n \sin(\pi/n)$, with n tending to infinity, which can be resolved using the basic trigonometric limit $\lim_{\theta \to 0}(\sin\theta)/\theta = 1$. Finally, these limit processes generalize the calculus of the area of the circle as a limit of the areas of circumscribed polygons. In fact, this limit occurs in the central cross section of the doublecake.

References

1. D. W. De Temple, An Archimedean property of the bicylinder, *College Math. J.* **25** (1994) 312–314.
2. A. J. Lo Bello, The volumes and centroids of some famous domes, *Math. Mag.* **61** (1988) 164–170.

Pythagorean Triples with Square and Triangular Sides

Sharon Brueggeman (sharon-brueggeman@utc.edu), University of Tennessee at Chattanooga, Chattanooga, TN 37403

Fermat [**2**] proved there are no Pythagorean triples in which the two smaller numbers (or legs) are both squares. On the other hand, Sierpinski [**3**] proved there are infinitely many in which both legs are consecutive triangular numbers. We begin this note by considering triples with one leg of each type, an example of which is (3, 4, 5) where $3 = t(2)$. (The nth triangular number is $t(n) = n(n+1)/2$.)

The triple (5, 12, 13) does not have our property. Yet if we multiply it by 3, the squarefree part of $12 = 2^2 \cdot 3$, we get (15, 36, 39) where $15 = t(5)$ and $36 = 6^2$. In general, we will take any Pythagorean triple with a square leg and multiply all three numbers by an appropriate square (1 in this example) so that the other leg becomes triangular.

