Here $\binom{N}{n} = 21$. The first factors of the terms in the sum on the left side of (4) start with $\binom{N}{1} = 7$ and progress to the right along the row for $N = 7$. The second factors of these terms are raised to the kth power and begin with $\binom{N-1}{2} = 15$ and progress up the column for $n = 2$. The numerical values for the identities in this example for $k = 1, 2$, and 3 are given below.

$$7 \cdot 15 - 21 \cdot 10 + 35 \cdot 6 - 35 \cdot 3 + 21 \cdot 1 = 21$$
$$7 \cdot 15^2 - 21 \cdot 10^2 + 35 \cdot 6^2 - 35 \cdot 3^2 + 21 \cdot 1^2 = 21^2$$
$$7 \cdot 15^3 - 21 \cdot 10^3 + 35 \cdot 6^3 - 35 \cdot 3^3 + 21 \cdot 1^3 = 21^3$$

The number of identities in the family is determined by how small n is relative to N. For example, if $N = 30$ and $n = 4$, then (4) holds for $k \leq 7$. The general relationship of each identity with respect to Pascal's triangle is the same as in the example. The first and second factors for the terms on the left side of (4) are found by starting with $\binom{N}{1}$ and moving to the right for the first factor and starting with $\binom{N-1}{2}$ and moving upwards for the second factor. The extensive literature on binomial coefficients has identities similar to the case where $k = 1$:

$$\sum_{j=1}^{N-n} (-1)^{j+1} \binom{N}{j} \binom{N-j}{n} = \binom{N}{n}.$$

For example, the reader is referred to the first chapter of Riordan's classic book, *Combinatorial Identities*. However, these identities do not have terms where factors are raised to an arbitrary power k as is the case in (4). The identity (4) is interesting in that it holds for all positive integers less than N/n. This allows us to write identities that hold for any number of consecutive integers but not beyond. For example, if $N = 1000$ and $n = 10$, then (4) holds for all $k \leq 99$ but not for any values beyond 99.

References

1. J. Riordan, *Combinatorial Identities*, Wiley, 1968.

Controlling the discrepancy in marginal analysis calculations

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Despite technology, we professors still love our little tricks in designing shortcuts and problems involving "nice numbers" that lead to easily predictable outcomes. Here's one such shortcut that I discovered recently.

Consider the typical case of a quadratic cumulative-cost function, often encountered in Calculus I and "Business Calculus" as a differentiation application. Here a hypothetical business produces x "widgets" for a total cost of $C(x) = ax^2 + bx + c$. (In order for $C(x)$ to be increasing, we restrict our attention to $0 \leq x \leq -b/2a$ with

$a < 0, b > 0$, and $c > 0$.) At production level x, the cost of the next item (item number x + 1) is $C(x+1) - C(x)$. This is the *marginal cost* of that item, sometimes called the marginal cost at level x. Inasmuch as

$$C(x+1) - C(x) = \frac{C(x+1) - C(x)}{1} \approx C'(x),$$

textbooks often define the marginal cost to be $C'(x)$ instead of that being an approximation. Regardless of which is definition and which is theorem, how do they differ for a quadratic function?

Easy!

$$\begin{aligned}
&\left|[C(x+1) - C(x)] - C'(x)\right| \\
&\quad = \left|a(x+1)^2 + b(x+1) + c - (ax^2 + bx + c) - (2ax + b)\right| \\
&\quad = |a|,
\end{aligned}$$

which gives us a nice simple result. (In the linear case, when $a = 0$, the two expressions of course give the same marginal cost at every production level.) So, if you want your two computations of marginal cost ($C(x+1) - C(x)$ and $C'(x)$) to be within 2 cents of one another, simply choose a quadratic model with $a = -0.02$. For example, if $C(x) = -0.02x^2 + 100x + 100000$ represents the cumulative cost of producing x DVD-players, then at any permissible production level x, the results of calculating marginal cost by the two methods will always differ by precisely 2 cents.

In closing, we note that this observation can be tied to other ideas in calculus and differential equations, such as the Mean Value Theorem. For instance, consider the question of which functions f satisfy

$$\frac{f(b) - f(a)}{b - a} = f'\left(\frac{a+b}{2}\right)$$

for arbitrary a and b; that is, the point of attainment of the theorem is always the midpoint of the interval. The answer is the set of all linear and quadratic polynomials. Of course, we don't usually teach the Mean Value Theorem in business calculus courses, not even for quadratic functions. Nevertheless, instructors might welcome the following additional observation: For a quadratic function $C(x)$, $C(x+1) - C(x) = C'(x + \frac{1}{2})$. This gives still another way of thinking about the discrepancy, namely, as $\left|C'(x + \frac{1}{2}) - C'(x)\right|$.

Stirling's formula via Riemann sums

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Introduction. Recently Rzadkowski [**5**] used Riemann sums in a novel way to prove Stirling's famous asymptotic approximation to the factorial

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n,$$

