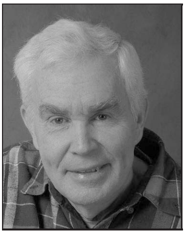
A New Method of Trisection

David Alan Brooks



David Alan Brooks was born in South Africa, where he qualified as an Air Force pilot before gaining a BA degree in classics and law, followed by an honors degree with a dissertation on ancient technology. To his regret, despite credits in sundry subjects, he has none in mathematics. After completing a second post-graduate honors degree in business administration and personnel management, he served as the deputy director of a training centre before becoming a full-time inventor. He has been granted numerous international patents. After having lived for six years on a boat, he now lives in an old chapel on a Welsh island. Thanks largely to a share of the royalties earned from two automatic poolcleaners manufactured in California, he is able to spend his days dreaming up additional geometry-intensive configurations of his revolutionary concept for efficient ultra-high-speed rotary "PHASE" engines, writing definitions for the all-limerick web dictionary OEDILF, and feeding ducks on a sacred lake once revered by Druids—also keen geometers (and perhaps frustrated trisectors) until they were crushed by the Romans in 60 AD.

He won wings as a proud Air Force flyer,
And strove, by degrees, to climb higher.
Reduced to a tent,
Then a boat (to invent),
He concluded that chapels are drier.

I do not, Archimedes, know why
You put marks on your ruler. Sir, I
Know that trisection tricks
Give a trisector kicks,
But why fudge when it's easy as pi?

One of the most interesting episodes in the long saga of angle trisection (see Dudley [**1**]) was a discovery made by Archimedes of Syracuse nearly 2300 years ago. He found that he could trisect an arbitrary angle simply and exactly if he made two little marks on the edge of his ruler. Taking this idea a step further, he showed that he could achieve the same feat by using virtual marks rather than physical ones. These were provided by the points of his compass as he held them alongside the ruler. Needless to say, both methods violate the strict straightedge-and-compass constraints of Euclidean construction, complete obedience to which makes trisection impossible (even though untold thousands of would-be trisectors have famously refused to believe this).

Archimedes' method

We show Archimedes' construction in Figure 1, where $\angle ABC$ is the angle to be trisected. Assuming that $AB = BC$, we draw a semicircle with centre B and radius AB. With the compass still open to this radius, we position its legs to form the virtual points

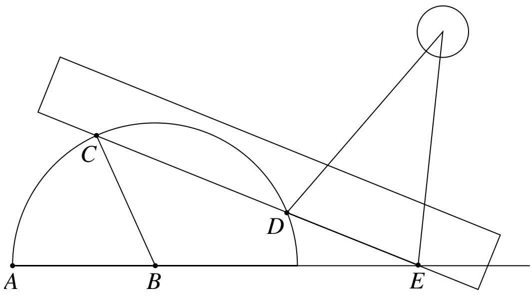


Figure 1. Archimedes' method.

D and E on the straightedge. We now slide this ruler and compass combination until D is on the semicircle, E is on the extension of AB, and the edge of the ruler passes though C. Then $\angle AEC$ is exactly one third of $\angle ABC$. (Proving this is left to the reader.)

For over two millennia those who have wished to trisect an arbitrary angle with only a ruler and compass have had to make these Archimedean marks—either real or virtual—on the edges of their rulers. Is this absolutely necessary? Surprisingly, the answer is *no*. We now show you one of several simple methods developed by the author, who had been alerted to the problem by Martin Gardner [**2**, Ch. 19], for trisecting an angle using only a compass and an *unmarked ruler*. However, in the time-honored tradition forced on trisectors by the constraints of Euclidean geometry, we do need to cheat a little. This will be done by taking advantage of the fact that real rulers and straightedges inevitably have ends.

A New Method

The angle to be trisected is the acute angle $\angle ABC$ of Figure 2. Let D be the midpoint of $\overline{BC}$. Draw an arc, with centre C, passing through D. Let E be the foot of the per-

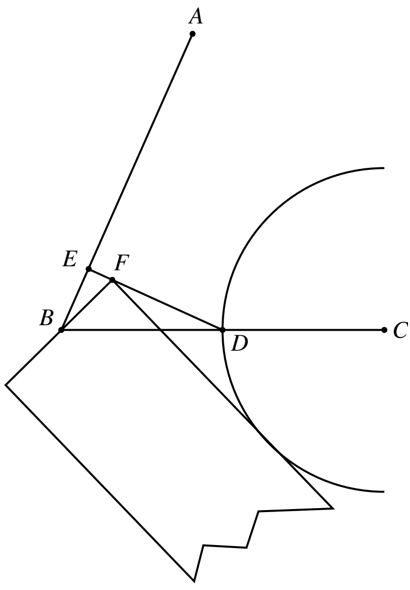


Figure 2. New method.

pendicular drawn from D to AB. Slide the ruler until its end is on B, its corner on the line $\overleftrightarrow{DE}$, say at F, and its edge tangent to the arc. Then $\overrightarrow{BF}$ will be a trisector of $\angle ABC$.

In proving that this is truly an exact trisection, we use the augmented diagram shown in Figure 3. Letting $\alpha = \angle ABC$, $\beta = \angle EBF$, and $\gamma = \angle FBC$, we show that $\gamma = 2\beta$. The point at which the edge of the ruler crosses $\overline{BC}$ we call G. The point at which it is tangent to the circle we call H. Let I denote the point at which a line through F, parallel to $\overleftrightarrow{BC}$, meets the extension of HC. For convenience, we assume that $CD = 1$. Observe that ΔBFG is similar to ΔCHG, so $\angle GCH = \gamma$, and hence $\tan\gamma = GH$. Also notice that $\angle FIH = \gamma$, so from ΔFIH we see that $\sin\gamma = FH/2$. Consequently, $FG = 2\sin\gamma - \tan\gamma$. We further note from ΔCGH that $\sec\gamma = CG$, so $DG = \sec\gamma - 1$. Now, applying the law of sines to ΔDFG, we get

$$\frac{\sin\angle DFG}{DG} = \frac{\sin\angle FDG}{FG}.$$

Because $\angle DFG = \beta$ and $\angle FDG = 90 - \alpha$, we have

$$\frac{\sin\beta}{\sec\gamma - 1} = \frac{\cos\alpha}{2\sin\gamma - \tan\gamma}$$

(since $\alpha = \beta + \gamma$). Thus, with an application of both algebra and trigonometry out of keeping with the extreme simplicity of the method itself, we arrive at the conclusion that $\cos\beta = \cos(\gamma - \beta)$. From this it follows that $\gamma = 2\beta$, and hence that $\overrightarrow{BF}$ trisects $\angle ABC$.

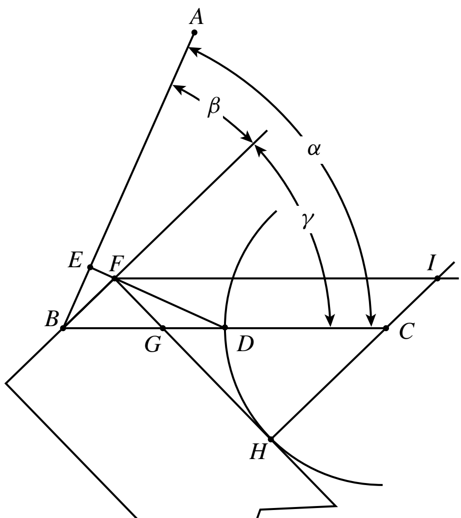


Figure 3. New method augmented.

In future, when mathematicians say that it is impossible to trisect arbitrary angles using only a compass and ruler, they should take care to mention not only the prohibition of Archimedes' ruler-and-compass collaboration, but should stress also that the ruler must not have an end. Finally, they should bear in mind that specialized trisecting

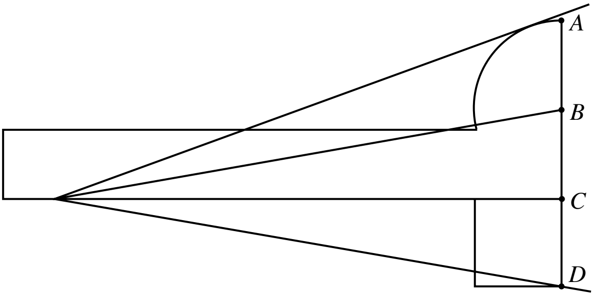


Figure 4. Tomahawk method.

tools such as the Tomahawk (see Figure 4) offer a service that can be provided just as well now by the humble ruler itself or a right triangle.

Like its namesake once swung by the Sioux,
The old Tomahawk trisector's through,
For it seems it's been matched
By the ruler unscratched,
Which can paddle its own smart canoe.

Acknowledgments. The author wishes to thank William Kahan of UCLA for providing the first independent proof of our result (on which our proof is based), and Cristobal Lemus Vidales and John Brooks-Jung, also of UCLA, for their corroborating proofs. Most of all, he wishes to thank the editor of CMJ for his patient guidance in the preparation of this paper.

References

1. Underwood Dudley, *The Trisectors*, Mathematical Association of America, 1994.
2. Martin Gardner, *Mathematical Carnival*, Knopf, 1975.

From Richard Syverson (rsyverson@comcast.net):

The question below appeared in a "Mystery category" in a quiz called "Where in the world are we?" in the *Minneapolis Star Tribune* on January 1, 2007.

Name the U.S. states that are rectangles.

For their surprising answer, see page 105.