$$
\begin{array}{rrrr|rrr}
 & & & & M & N & \\
1 & -3 & -5 & 4 & 1 & -2 & \\
1 & -1 & 2 & & 1 & -1 & 2 \\
\hline
 & -2 & -7 & 4 & & & \\
 & -2 & 2 & -4 & & & \\
\cline{2-4}
 & & -9 & 8 & & & \\
 & & K & L & & &
\end{array}
$$

Substituting the coefficients in (2) (note that $t - (n-1)/2 = 1$) gives

$$\frac{x^5 - 4x^4 + 3x^2 - 2}{(x^2 - x + 2)^3} = \frac{x - 2}{x^2 - x + 2} + \frac{-9x + 8}{(x^2 - x + 2)^2} + \frac{14x - 10}{(x^2 - x + 2)^3}.$$

Note also that $x^2 - x + 2 = (x - 1/2)^2 + 4/7$. On the right hand side of the above expression, replacing the coefficients -2, 8, and -10 in the numerators by $-2 + 1/2M$, $8 + 1/2M$, and $-10 + 1/2M$, respectively, we get

$$\frac{x^5 - 4x^4 + 3x - 2}{((x - 1/2)^2 + 7/4)^3} = \frac{(x - 1/2) - 3/2}{(x - 1/2)^2 + 7/4} + \frac{-9(x - 1/2) + 7/4}{((x - 1/2)^2 + 7/4)^2}$$
$$+ \frac{14(x - 1/2)^2 - 3}{((x - 1/2)^2 + 7/4)^3}.$$

This last expression is an easily antidifferentiable form.

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An Elegant Mode for Determining the Mode

D.S. Broca (dsbroca@iimk.ac.in), Indian Institute of Management Kozhikode, Kozhikode 673 571, Kerala, India

For any probability distribution, the mode, like the mean and median, is a measure of central tendency. Geometrically, it represents the relative maximum of the probability density function (pdf) and thus is the most striking feature in the curve's topography. In locating the position of the modal value, the classical first or second derivative tests can prove quite tedious since a density function is usually the product of several factors involving x. The purpose of this paper is to demonstrate how, instead of using the original function $f(x)$, the alternative $g(x) = \ln f(x)$ can be employed, often leading to considerable simplifications.

The mode is important in its own right as a descriptive parameter. For a unimodal distribution, this value can, especially in cases where the mean does not exist, be taken as a measure of central tendency. The position of the mode also helps measure skewness (or, lack of symmetry) of the curve. The Pearson coefficient of skewness, $sk_p = (mean - mode)/\sigma \simeq 3(mean - median)/\sigma$, depends upon the relative position of the mean μ, the median μ_e, and the mode m_o. For a positively skewed distribution, $m_o < \mu_e < \mu$, that is, $sk_p > 0$, while for a negatively skewed distribution, the inequality is reversed.

The method

The following method of finding the mode was originally proposed by Rahman [**3**, Ch. 7]. We revisit it here, primarily to illustrate its effectiveness as a teaching tool; in addition, it is hoped that the method will become a more standard part of textbooks on probability and statistics. We assume without loss of generality that $f(x) > 0$. Then, by taking natural logarithms, $g(x) = \ln f(x) \Longleftrightarrow f(x) = e^{g(x)}$. Differentiation yields $f'(x) = e^{g(x)}g'(x)$, so $f'(x_0) = 0$ if and only if $g'(x_0) = 0$ since the exponential function is always positive. Differentiating again gives $f''(x) = e^{g(x)}g''(x) + e^{g(x)}[g'(x)]^2$. Evaluating the second derivative at the critical point x_0, we get $f''(x_0) = e^{g(x_0)}g''(x_0)$. Therefore, $f''(x_0) < 0$ if and only if $g''(x_0) < 0$. Thus,

$$\left.\begin{matrix} f'(x_0) = 0 \\ f''(x_0) < 0 \end{matrix}\right\} \Longleftrightarrow \left\{\begin{matrix} g'(x_0) = 0 \\ g''(x_0) < 0. \end{matrix}\right.$$

It is clear that the two operations of applying the second derivative test on the original function $f(x)$and applying it on $g(x) = \ln f(x)$ are mathematically equivalent. Which of the two operations is easier? Using $g(x)$ to compute the mode essentially presents two main advantages. First, a density function is usually the product of several factors. Taking natural logarithms converts products into sums, which are easier to differentiate. In this vein, the method is akin to logarithmic differentiation typically taught in first semester calculus. In fact, logarithmic differentiation is routinely employed in finding maximum likelihood estimates by using log-likelihoods (see, for example, [**1**, Ch. 6]). This paper applies the same idea to finding the mode. Second, many density functions involve the exponential function, an elementary transcendental. The natural logarithm, being its inverse, leads to the "cancellation" effect, and this results in polynomials, which are easier to differentiate.

Examples

We illustrate the procedure with three positively skewed densities: extreme value (Type I), Weibull, and lognormal (see [**2**, Ch. III] for a general discussion). Of course, the same procedure is applicable to symmetric pdfs that satisfy the condition $f(\mu + x) = f(\mu - x)$ for symmetry about the point $x = \mu$ (for example, Gaussian, Laplace, Cauchy, and logistic distributions). However, in these cases, where the above condition is clearly fulfilled, the mean, median, and mode coincide; hence, the mode is obviously located at $x = \mu$, which eliminates the need for further computation.

Extreme value density The limiting distribution of the greatest (or least) value in ordered random samples has found application in studying such diverse phenomena as flood flows, earthquakes, rainfall, corrosion, and even stock price movements. Extreme value distributions generally belong to one of three families, called Types 1, 2, and 3. Of these, Type 1 is by far the most common, referred to by some authors as "the" extreme value distribution. The extreme value (Type I) distribution (also called Gumbel) has a pdf given by

$$f(x) = \theta^{-1} e^{-(x-\mu)/\theta} \exp(-e^{-(x-\mu)/\theta}) \qquad -\infty < x < \infty,$$

where $\mu \in R^1$ is the location parameter and $\theta > 0$ is a scale parameter. Now,

$$g(x) = \ln f(x) = -\ln\theta - (x-\mu)/\theta - e^{-(x-\mu)/\theta}.$$

Taking the first derivative, we find

$$g'(x) = -\theta^{-1} + \theta^{-1} e^{-(x-\mu)/\theta},$$

so the only critical value is at $x = \mu$. The second derivative of $g(x)$ is

$$g''(x) = -\theta^{-2} e^{-(x-\mu)/\theta},$$

which is negative at $x = \mu$. Thus, the extreme value (Type I) distribution has a single mode located at $x = \mu$.

Weibull distribution This distribution is named after Waloddi Weibull, a Swedish physicist who used it to represent the breaking strength of materials. It has found widespread use in reliability and quality control studies. From a broader perspective, the Weibull arises as a special case of a Type 3 extreme value distribution. A random variable X is defined to have a Weibull distribution if its pdf is given by

$$f(x) = c/a(x/a)^{c-1} e^{-(x/a)^c} \qquad x > 0,$$

where $a > 0$ is a scale parameter and $c > 0$ is a shape parameter. For $c = 1$, the distribution degenerates into the negative exponential with $\lambda = 1/a$. For $c < 1$, there is a vertical asymptote at $x = 0$. We consider the case $c > 1$.

$$g(x) = \ln f(x) = \ln(c/a) + (c-1)\ln(x/a) - x^c/a^c.$$

Taking the first derivative gives

$$g'(x) = (c-1)/x - cx^{(c-1)}/a^c,$$

and we locate the critical value at $x = a[(c-1)/c]^{1/c}$. Differentiating again gives

$$g''(x) = -\frac{c-1}{x^2} - \frac{c(c-1)x^{c-2}}{a^c},$$

which is clearly negative when $x > 0$, $a > 0$, and $c > 1$. It may be noted that the negativity of the above expression is established directly based on restrictions on x, a, and c without actually substituting in the critical value itself. Furthermore, the negativity reveals the nature of the critical point at $x = a[(c-1)/c]^{1/c}$ as being the mode (maximum ordinate).

Lognormal density This density usefully represents the distribution of size for varied kinds of 'natural' economic or physical units. The case of the lognormal requires somewhat more mathematical manipulation than the previous two examples. Imagine these computations when using the original function $f(x)$! The pdf of the lognormal is

$$f(x) = \frac{1}{\sigma x\sqrt{2\pi}} e^{-(\ln x - \mu)^2/2\sigma^2} \qquad x > 0,$$

where $\mu \in R^1$and $\sigma > 0$ are the location and scale parameters. Now, by taking the natural logarithm of $f(x)$ in the customary fashion, we get

$$g(x) = \ln f(x) = -\ln\sigma - \ln x - \ln\sqrt{2\pi} - (\ln x - \mu)^2/2\sigma^2.$$

Differentiation yields

$$g'(x) = -\frac{1}{x} - \frac{(\ln x - \mu)}{\sigma^2 x},$$

from which we find that the critical value is $x = e^{\mu-\sigma^2}$. Differentiating again, we get

$$g''(x) = \frac{1}{x^2} - \frac{1}{\sigma^2}\left\{\frac{1 - \ln x + \mu}{x^2}\right\},$$

which is negative at the critical value. This confirms the existence of the mode at $x = e^{\mu-\sigma^2}$. In conclusion, we note that the method given here can also be used to locate inflection points of the density curve by solving the equation

$$f''(x) = e^{g(x)}g''(x) + e^{g(x)}[g'(x)]^2 = 0.$$

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Searching for Möbius

Al Cuoco (acuoco@edc.org), Education Development Center, Newton, MA 02158

The Möbius function μ, defined on the integers by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \text{ is divisible by a square} \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes,} \end{cases}$$