REMARK. The equations $n = 0$ and $n = 1$ are satisfied by $A = B = C = D = \frac{25}{24}$ giving the equal-weight formula for 0-ended cubics

$$\int_a^b f(x)\,dx = \frac{25}{24} h\,[f(a+h) + f(a+2h) + f(a+3h) + f(a+4h)]$$

where $h = \frac{1}{5}(b-a)$.

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A Short Proof of the Two-sidedness of Matrix Inverses

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In this note we offer a short proof that for any $n \times n$ matrices A and C over a field of scalars,

$$AC = I \text{ if and only if } CA = I.$$

An interesting aspect of this biconditional is the fact that it is equivalent to the conditional if $AC = I$ then $CA = I$; one simply interchanges the roles of A and C.

We assume the reader is familiar with the reduction of a matrix to row-echelon form. Recall that if A is $n \times n$ matrix then its reduced row-echelon form is a matrix of the same size with zeros in the pivot columns except for the pivots which are equal to 1. It is achieved by applying elementary row operations (row swapping, row addition, row scaling) to A. An elementary matrix is one obtained by applying a single elementary row operation to the $n \times n$ identity matrix I. Elementary matrices have inverses that are also elementary matrices. Left multiplication of A by an elementary matrix E effects the same row operation on A that was used to create E.

Let H be the reduced row echelon form of A, and let P be the product of those elementary matrices (in the appropriate order) that reduce A to H. P is an invertible matrix such that $PA = H$. Notice that H is the identity matrix if and only if it has n pivots.

The proof. Beginning with $AC = I$, we left multiply this equation by P obtaining $PAC = P$ or $HC = P$. If H is not the identity matrix it must have a bottom row of zeros forcing P to have likewise a bottom row of zeros, and this contradicts the invertibility of P. Thus $H = I$, $C = P$, and the equation $PA = H$ is actually $CA = I$.

This argument shows at once that (i) a matrix is invertible if and only if its reduced row echelon form is the identity matrix, and (ii) the set of invertible matrices is precisely the set of products of elementary matrices.

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Root Preserving Transformations of Polynomials

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Consider the real vector space $\mathcal{P}_2$ of all polynomials of degree at most 2. High-school students study the roots of the polynomials in $\mathcal{P}_2$, while linear algebra students study linear transformations on $\mathcal{P}_2$. Is it possible to bring these two groups together to do some joint research?

For example, a linear algebra student chooses a specific linear transformation $T : \mathcal{P}_2 \to \mathcal{P}_2$ and asks others to study the roots of a polynomial

$$p(x) = ax^2 + bx + c, \quad x \in \mathbb{R},$$

and the roots of its image

$$(Tp)(x) = bx^2 + cx + a, \quad x \in \mathbb{R}. \tag{1}$$

Here a, b, and c are arbitrary real numbers. The students may immediately notice that the polynomial $x^2 + x + 1$ is unchanged by this transformation. Hence this particular polynomial and its image have the same (complex) roots. After some "trial and error," a high-school student points out that the polynomial $x^2 + 3x + 2$ has the roots -1 and -2, while its image $3x^2 + 2x + 1$ does not have real roots. Their next interesting discovery is that, with $v \neq 1$, the polynomial $x^2 + (v-1)x - v$ has roots 1 and $-v$, while its image $(v-1)x^2 - vx + 1$ has roots 1 and $1/(v-1)$. This is curious since in this case a polynomial and its image have one common root, namely 1.

After further study the students conclude that there doesn't seem to be any general simple relationship between the roots of a polynomial p and the roots of its image Tp under the linear transformation given by (1). But the obvious fact is that there are plenty of other linear transformations on $\mathcal{P}_2$; will it always be the case that there is no simple relationship between the roots? Clearly, a non-zero multiple of the identity on