Finally, to see that Theorem 3 is sharp, we again consider the case when $0 < \varepsilon < h$, but this time we put

$$f_\varepsilon(x) = \begin{cases} \frac{x^5}{30\varepsilon^2} & \text{for } 0 \le x \le \varepsilon, \\ \frac{x^3}{3} - \frac{2\varepsilon x^2}{3} + \frac{\varepsilon^2 x}{2} - \frac{2\varepsilon^3}{15} & \text{for } \varepsilon < x \le h, \end{cases}$$

and extend f_ε to be an even function. Another straightforward but tedious argument shows that f_ε is three times continuously differentiable and $\left|f_\varepsilon'''(x)\right| \le 2$ for all $x \in [-h, h]$. In this case $M = 2$ and our estimate in Theorem 3 becomes $\frac{h^4}{18}$; a direct calculation shows $\frac{h^4}{18} - |E_2| \le O(\varepsilon^2)$. Note that in [**4**], Talman shows that $|E_2| \le \frac{2Mh^4}{9}$. This is similar in form to our result but not sharp as Talman himself points out.

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The Class of Heron Triangles

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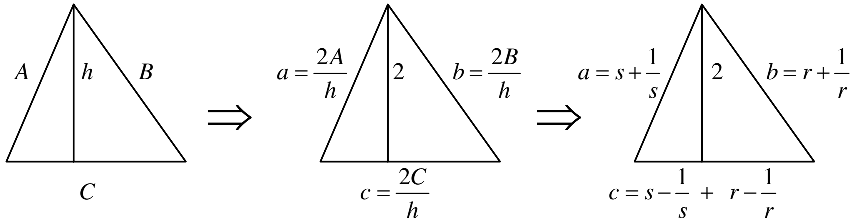
A Heron triangle is one with rational sides and rational area. A glance at sources ranging from Dickson's wonderful compilation [**1**] to the modern Wolfram website [**2**] indicates that such triangles have not only been but will continue to be a fascination. If a Heron triangle is scaled up using the lowest common multiple of the denominators of the sidelengths, then the triangle is similar to one having integer sides and integer area. Three integer parameters m, n, k can then be given that generate all such Heron triangles: the sides have the form (see [**2**])

$$n(m^2 + k^2),\ m(n^2 + k^2),\ (m + n)(mn - k^2).$$

In this note we start with a Heron triangle and scale it (usually down) to produce a Heron triangle with altitude 2. We are then able to obtain two rational parameters that generate this family of Heron triangles.

THEOREM. *After scaling, every Heron triangle has the form of the right-most triangle in the figure below, for rationals r and s with $r, s > 1$.*

Proof. Any triangle can be oriented so that it has the configuration of the first triangle in the figure. If we assume that it is a Heron triangle and if we multiply its sides A, B, C and its (rational) altitude h by the scaling factor $2/h$, we obtain the second triangle in the figure with rational sides a, b, c and altitude 2.



Since $a, b > 2$ there are real numbers $r, s > 1$ such that $a = s + 1/s$ and $b = r + 1/r$. One simply solves the equations $s^2 - as + 1 = 0$ and $r^2 - br + 1 = 0$, obtaining

$$s = \frac{a + \sqrt{a^2 - 4}}{2}, \quad r = \frac{b + \sqrt{b^2 - 4}}{2}.$$

Now c is sum of the bases of the two right triangles in the figure (third triangle); using the Pythagorean theorem twice we see that $c = s - 1/s + r - 1/r$. It remains to show that both r and s are rational. But this follows from the equations

$$\begin{aligned} \frac{a+b+c}{2} &= r + s, \\ \frac{a+b-c}{2} &= \frac{1}{r} + \frac{1}{s}, \\ \frac{a-b+c}{2} &= s - \frac{1}{r}, \\ \frac{-a+b+c}{2} &= r - \frac{1}{s}. \end{aligned}$$

Dividing the first equation by the second shows that rs is rational; the third equation then shows that r is rational, and s follows suit because of the fourth equation. ■

The interested reader is invited to show how the (r, s) parameters yield the (m, n, k) parameters.

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