Integral Solutions to the Equation $x^2 + y^2 + z^2 = u^2$: A Geometrical Approach

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In his *History of the Theory of Numbers*, Dickson [4] cites six identities, each of which gives infinitely many integral solutions of the equation

$$x^2+y^2+z^2=u^2. \qquad (*)$$

These are:

(1) $(2pr)^2+(2qr)^2+(p^2+q^2-r^2)^2=(p^2+q^2+r^2)^2$,
by V. A. Lebesgue [7];

(2) $[p(p+q)]^2+[q(p+q)]^2+(pq)^2=(p^2+pq+q^2)^2$,
by U. Dainelli [3];

(3) $[2qr(m^2-n^2)]^2+[(m^2-n^2)(q^2-r^2)]^2+[2mn(q^2+r^2)]^2=[(m^2+n^2)(q^2+r^2)]^2$,
by C. Gill [6];

(4) $(4mp)^2+[(m^2-1)(p^2+1)]^2+[2m(p^2-1)]^2=[(m^2+1)(p^2+1)]^2$,
by J. A. Euler [5];

(5) $(4m^2n^2)^2+(m^4-n^4)^2+[2mn(m^2-n^2)]^2=[(m^2+n^2)^2]^2$,
by the Japanese Matsunango [8]; and

(6) $q^2+(q+1)^2+[q(q+1)]^2=(q^2+q+1)^2$,
by P. Cossali [2].

What we would like to do here is to show that these identities are special cases of a more general identity, by using analytical geometry of three dimensions.

If we divide equation $(*)$ by u^2, we get $(x/u)^2+(y/u)^2+(z/u)^2=1$. Now if x, y, z, and u are integers, then x/u, y/u, z/u are rational numbers, and our problem reduces to that of finding triples of rational solutions (x', y', z') to the equation

$$x'^2+y'^2+z'^2=1. \qquad (7)$$

The graph of this equation in E^3 is a sphere, and for m, $n\in Z$, the point

$$\left(\frac{m^2-n^2}{m^2+n^2}, \frac{2mn}{m^2+n^2}, 0\right) \qquad (8)$$

lies on the sphere and represents a solution to equation (7). Of course $((m^2-n^2)/(m^2+n^2)$, $2mn/(m^2+n^2))$ is the well-known solution to the equation $x'^2+y'^2=1$. There are many derivations of this solution, one of them by H. Wright [9], in which he used plane coordinate geometry.

Now consider the line in E^3 through the point in (8), whose direction is given by the vector (r, p, q). The parametric equations for this line are:

$$x'=\frac{m^2-n^2}{m^2+n^2}+rt, \qquad y'=\frac{2mn}{m^2+n^2}+pt, \qquad z'=qt,$$

where $r, p, q\in Z$, and t is a parameter. Substituting for x', y' and z' in equation (7) gives

$$\left(\frac{m^2-n^2}{m^2+n^2}+rt\right)^2+\left(\frac{2mn}{m^2+n^2}+pt\right)^2+(qt)^2=1,$$

and solving for the parameter t, we find that $t=0$ or

$$t = \frac{-2r(m^2 - n^2) - 4mnp}{(m^2 + n^2)(p^2 + q^2 + r^2)}. \tag{9}$$

The value $t = 0$ corresponds to the point in (8), and the second value of t, given by (9), corresponds to the other point at which the straight line intersects the surface of the sphere. Using the value of the parameter t in (9), and denoting by u its denominator, we get a new solution of equation $(*)$:

$$\begin{aligned} x &= (m^2 - n^2)(p^2 + q^2 - r^2) - 4mnpr \\ y &= 2mn(r^2 - p^2 + q^2) - 2rp(m^2 - n^2) \\ z &= -2qr(m^2 - n^2) - 4mnpq \\ u &= (m^2 + n^2)(p^2 + q^2 + r^2). \end{aligned} \tag{10}$$

And now the following substitutions in (10) yield the identities cited in our opening paragraph:

(1) $m = 1, \quad n = 0.$
(2) $m = 1, \quad n = 0, \quad r = p + q.$
(3) $p = 0.$
(4) $r = 0, \quad n = q = 1.$
(5) $r = 0, \quad p = m, \quad q = n.$
(6) $m = p = 1, \quad n = 0, \quad r = 1 + q.$

Although the identity derived here provides more solutions to equation $(*)$ than any of the six given identities, it still does not give *all* the solutions of $x^2 + y^2 + z^2 = u^2$. For example, $u = 27$ is representable as $(m^2 + n^2)(p^2 + q^2 + r^2)$ in two ways; either $m = 1,\ n = 0,\ p = q = r = 3$ or $m = 3,\ n = 0,\ p = q = r = 1$. The solution (10) to equation $(*)$ in either case is given by $18^2 + 18^2 + 9^2 = 27^2$. However there are other solutions to $(*)$ for $u = 27$:

$$\begin{aligned} 23^2 + 14^2 + 2^2 &= 27^2 \\ 26^2 + 7^2 + 2^2 &= 27^2 \\ 22^2 + 7^2 + 14^2 &= 27^2. \end{aligned}$$

These three solutions can be obtained from the identity

$$(p^2 + q^2 - r^2 - s^2)^2 + [2(pr + qs)]^2 + [2(ps - qr)]^2 = (p^2 + q^2 + r^2 + s^2)^2,$$

which provides the complete solution of $x^2 + y^2 + z^2 = u^2$ as noted by E. Catalan [**1**].

Applying the same technique and making use of these identities (or special cases of them), one can derive identities that generate integral solutions to the equation $x^2 + y^2 + z^2 + w^2 = u^2$ or even to the more general equation $x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2 = u^2$.

References

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