An Elementary Proof of Error Estimates for the Trapezoidal Rule

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Essentially every calculus textbook contains the trapezoidal rule for estimating definite integrals; this rule can be stated precisely as follows:

If f is continuous, then for each integer $n > 0$ the integral of f on $[a, b]$ is approximated by

$$T_n(f) = \frac{b-a}{2n}\big(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\big),$$

where $x_i = a + i(b-a)/n$, $0 \le i \le n$. Further, if f is twice differentiable, f'' is continuous, and $f''(t) \le M$ for $t \in [a, b]$, then

$$E_n^T(f) = \left| T_n(f) - \int_a^b f(t)\,dt \right| \le \frac{(b-a)^3}{12n^2} M. \tag{1}$$

(See, for instance, Stewart [**7**].) There are two problems, however, with this result. First, calculus books generally omit the proof, and instead refer the reader to an advanced text on numerical analysis. In such books the trapezoidal rule is usually derived as a corollary to a more general result for Newton-Cotes quadrature methods, and the proof, depending on polynomial approximation, is generally not accessible to calculus students. (See, for example, Ralston [**5**].)

Second, the error estimate given by (1) is not applicable to such well-behaved functions as $x^{3/2}$ and $x^{1/2}$ on [0, 1], since neither has a bounded second derivative. In other words, you can use the trapezoidal rule to approximate their integrals, but for a given n you have no idea, *a priori*, how good the approximation is.

In this note we give an elementary proof of inequality (1). The key idea is to use integration by parts "backwards." The argument is straightforward and should be readily understood by students in second-semester calculus. This approach is not new and goes back to Von Mises [**8**] and Peano [**4**]. (See also Ghizzetti and Ossicini [**3**].) However, our exact proof is either new or long forgotten—see Cruz-Uribe and Neugebauer [**2**] for a survey of the literature.

Our proof has two other advantages. First, we can omit the assumption that f'' is continuous, and replace it with the weaker assumption that it is (Riemann) integrable. Second, we can adapt our proof to give estimates for functions that do not have bounded second derivatives, such as $x^{3/2}$ and $x^{1/2}$.

Proving the inequality. The first step is to find an expression for the error on each interval $[x_{i-1}, x_i]$. If we divide the integral of f on $[a, b]$ into the sum of integrals on

each of these subintervals, we have

$$E_n^T(f) = \left| \sum_{i=1}^{n} \left(\frac{b-a}{2n}(f(x_{i-1}) + f(x_i)) - \int_{x_{i-1}}^{x_i} f(t)\,dt \right) \right|.$$

For each i, $0 \le i \le n$, define

$$L_i = \frac{b-a}{2n}(f(x_{i-1}) + f(x_i)) - \int_{x_{i-1}}^{x_i} f(t)\,dt. \tag{2}$$

Let c_i denote the center of the interval $[x_{i-1}, x_i]$: $c_i = (x_{i-1} + x_i)/2$. Then

$$x_i - c_i = c_i - x_{i-1} = \frac{b-a}{2n},$$

and if we apply integration by parts "backwards," we see that we can express the error in terms of the first derivative:

$$L_i = \int_{x_{i-1}}^{x_i} (t - c_i) f'(t)\,dt. \tag{3}$$

(Given (3), it is easy to apply integration by parts to show that (2) holds; it is more subtle to argue in the other direction.) If we integrate this result by parts, we can express the error in terms of the second derivative:

$$L_i = \frac{1}{2} \int_{x_{i-1}}^{x_i} \left(\left(\frac{b-a}{2n} \right)^2 - (t - c_i)^2 \right) f''(t)\,dt. \tag{4}$$

Therefore, if $|f''(t)| \le M$ for $t \in [a, b]$, we have that

$$\begin{aligned} E_n^T(f) &\le \sum_{i=1}^{n} |L_i| \\ &\le \frac{1}{2} \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \left| \left(\frac{b-a}{2n} \right)^2 - (t - c_i)^2 \right| |f''(t)|\,dt \\ &\le \frac{M}{2} \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \left(\left(\frac{b-a}{2n} \right)^2 - (t - c_i)^2 \right) dt. \end{aligned}$$

For each i, a straightforward calculation shows that

$$\int_{x_{i-1}}^{x_i} \left(\left(\frac{b-a}{2n} \right)^2 - (t - c_i)^2 \right) dt = \frac{1}{6} \left(\frac{b-a}{n} \right)^3.$$

Hence,

$$E_n^T(f) \le \frac{M}{2} \sum_{i=1}^{n} \frac{1}{6} \left(\frac{b-a}{n} \right)^3 = \frac{(b-a)^3}{12n^2} M,$$

which is precisely what we wanted to prove.

What if the second derivative is not bounded? The heart of the proof of (1) is using (4) to estimate the error. However, a very similar argument works if we start

with (3). Suppose $|f'(t)| \le N$ for $t \in [a, b]$. Then

$$|L_i| \le N \int_{x_{i-1}}^{x_i} |t - c_i|\, dt = \frac{N}{4}\left(\frac{b-a}{n}\right)^2,$$

which yields

$$E_n^T(f) \le \frac{(b-a)^2}{4n} N.$$

For example, if $f(x) = x^{3/2}$ on $[0, 1]$ then $|f'(x)| \le 3/2$, so

$$E_n^T(f) \le \frac{3}{8n}.$$

We can get a better estimate if the integrals of f'' and $|f''|$ exist as improper integrals, as is the case for $f(x) = x^{3/2}$. (Alternatively, we could use a more general definition of the integral, such as the Lebesgue integral or the Henstock-Kurzweiler integral. See [**1, 6**].) For in this case we can still apply integration by parts to derive (4). Since for each i,

$$0 \le \left(\frac{b-a}{2n}\right)^2 - (t - c_i)^2 \le \left(\frac{b-a}{2n}\right)^2,$$

it follows that

$$|L_i| \le \frac{1}{2}\left(\frac{b-a}{2n}\right)^2 \int_{x_{i-1}}^{x_i} |f''(t)|\, dt.$$

Hence,

$$E_n^T(f) \le \frac{(b-a)^2}{8n^2} \int_a^b |f''(t)|\, dt.$$

Again, if we let $f(x) = x^{3/2}$ on $[0, 1]$, this yields the estimate

$$E_n^T(f) \le \frac{3}{16n^2}.$$

Given a function such as $x^{1/2}$ on $[0, 1]$, whose first derivative is integrable as an improper integral, but whose second derivative is not, we can use the same argument to derive similar estimates from (3):

$$E_n^T(f) \le \frac{b-a}{2n} \int_a^b |f'(t)|\, dt.$$

Thus, if $f(x) = x^{1/2}$ on $[0, 1]$,

$$E_n^T(f) \le \frac{1}{2n}.$$

Further extensions. We prove these and related results for the trapezoidal rule in [**2**]. We also show that all of these error estimates are sharp: for each n we construct a function f such that equality holds.

We also use the same techniques to prove error estimates for Simpson's rule in terms of the first and second derivative. The classical error estimate for Simpson's rule involve the fourth derivative [**5, 7**], and it is an open problem to extend our ideas to give an elementary proof of this result.

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Triangles with Integer Sides

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It is well known [**1, 2, 3, 4, 5, 6**] that the number T_n of triangles with integer sides and perimeter n is given by

$$T_n = \begin{cases} \langle (n+3)^2/48 \rangle & \text{if } n \text{ is odd} \\ \langle n^2/48 \rangle & \text{if } n \text{ is even,} \end{cases}$$

where $\langle x \rangle$ is the integer closest to x. The object of this note is to give as quick a proof of this as I can, starting with:

LEMMA 1. *The number S_n of scalene triangles with integer sides and perimeter n is given for $n \geq 6$ by*

$$S_n = T_{n-6}.$$

Proof. If $n = 6, 7, 8$, or 10, both are 0. Otherwise, given a scalene triangle with integer sides $a < b < c$ and perimeter n, let $a' = a - 1$, $b' = b - 2$, $c' = c - 3$. Then a', b', c' are the sides of a triangle of perimeter $n - 6$. Moreover, the process is reversible, so the result follows. ■

COROLLARY. *$T_n - T_{n-6} = I_n$, where I_n denotes the number of isosceles (including equilateral) triangles with integer sides and perimeter n.*