Four Ways to Evaluate a Poisson Integral

HONGWEI CHEN
Christopher Newport University
Newport News, VA 23606
hchen@pcs.cnu.edu

In general, it is difficult to decide whether or not a given function can be integrated in elementary ways. In light of this, it is quite surprising that the value of the Poisson integral

$$I(x) = \int_0^\pi \ln(1 - 2x\cos\theta + x^2)\,d\theta$$

can be determined precisely. Even more surprising is that we can do so for every value of the parameter x. Using four different methods, we will show that

$$I(x) = \begin{cases} 0, & \text{if } |x| < 1; \\ 2\pi \ln|x|, & \text{if } |x| > 1. \end{cases}$$

Our integral is one of several known as the Poisson Integral; all are related in some way to Poisson's integral formula, which recovers an analytic function on the disk from its boundary values, a relationship we mention below. However, none of our methods involves complex analysis at all. The first one uses Riemann sums and relies on a trigonometric identity. The second method is based on a functional equation and involves a sequence of integral substitutions. The third method uses parametric differentiation and the half-angle substitution. We finish with an approach based on infinite series. It is interesting to see how wide a range of mathematical topics are exploited. These evaluations are suitable for an advanced calculus class and provide a very nice application of Riemann sums, functional equations, parametric differentiation, and infinite series.

We begin with three elementary observations:

1. $I(0) = 0$.
2. $I(-x) = I(x)$.
3. $I(x) = 2\pi\ln|x| + I(1/x), \quad (x \neq 0)$.

The reader can probably supply the proofs for these, but we will demonstrate the third. If $x \neq 0$, we have

$$\begin{aligned} I(x) &= \int_0^\pi \ln\left[x^2\left(1 - \frac{2}{x}\cos\theta + \frac{1}{x^2}\right)\right] d\theta \\ &= \int_0^\pi \ln x^2\,d\theta + I(1/x) = 2\pi\ln|x| + I(1/x). \end{aligned}$$

In view of this third observation, our main formula follows easily once we show that $I(x) = 0$ for $|x| < 1$. This will be the goal of the next four sections.

I. Using Riemann sums Since

$$1 - 2x\cos\theta + x^2 \geq (1 - |x|)^2, \quad \text{for } |x| < 1,$$

the integrand is continuous and integrable. Partition the interval $[0, \pi]$ into n equal subintervals by the partition points $x_k = k\pi/n$, for $1 \leq k \leq n$. The Riemann sum for $I(x)$, $\mathcal{R}_n$, can be simplified using laws of logarithms:

$$\begin{aligned} \mathcal{R}_n &= \frac{\pi}{n} \sum_{k=1}^{n} \ln\left(1 - 2x\cos\left(\frac{k\pi}{n}\right) + x^2\right) \\ &= \frac{\pi}{n} \ln\left[(1+x)^2 \prod_{k=1}^{n-1}\left(1 - 2x\cos\left(\frac{k\pi}{n}\right) + x^2\right)\right]. \end{aligned} \tag{1}$$

To proceed further, let $\omega = \exp(i\pi/n)$. The distinct roots of the polynomial $x^{2n} - 1$ are ω^k for $-n \leq k < n$, so

$$x^{2n} - 1 = \prod_{k=-n}^{n-1} (x - \omega^k).$$

Combining the conjugate factors and appealing to De Moivre's theorem, we find

$$\begin{aligned} x^{2n} - 1 &= (x^2 - 1) \prod_{k=1}^{n-1} (x - \omega^k)(x - \omega^{-k}) \\ &= (x^2 - 1) \prod_{k=1}^{n-1} \left(1 - 2x\cos\left(\frac{k\pi}{n}\right) + x^2\right), \end{aligned}$$

so that

$$\prod_{k=1}^{n-1} \left(1 - 2x\cos\left(\frac{k\pi}{n}\right) + x^2\right) = \frac{x^{2n} - 1}{x^2 - 1}. \tag{2}$$

Substituting the identity (2) into (1), we have

$$\mathcal{R}_n = \frac{\pi}{n} \ln\left(\frac{x+1}{x-1}(x^{2n} - 1)\right).$$

Since $|x| < 1$, $x^{2n} \to 0$ as $n \to \infty$. Hence, we obtain

$$I(x) = \lim_{n\to\infty} \mathcal{R}_n = \lim_{n\to\infty} \frac{\pi}{n} \ln\left(\frac{x+1}{x-1}(x^{2n} - 1)\right) = 0.$$

Remark This method relied on the trigonometric identity (2), which is interesting in its own right.

II. Using a functional equation The functional equation we have in mind is

$$I(x) = I(-x) = \tfrac{1}{2} I(x^2). \tag{3}$$

Adding the two integrals below and using laws of logarithms, we obtain

$$I(x) + I(-x) = \int_0^{\pi} \ln(1 - 2x^2\cos 2\theta + x^4)\, d\theta.$$

Setting $\alpha = 2\theta$ gives

$$\begin{aligned} I(x) + I(-x) &= \frac{1}{2}\int_0^{2\pi} \ln(1 - 2x^2\cos\alpha + x^4)\,d\alpha \\ &= \frac{1}{2}I(x^2) + \frac{1}{2}\int_\pi^{2\pi} \ln(1 - 2x^2\cos\alpha + x^4)\,d\alpha. \end{aligned}$$

The substitution $\alpha = 2\pi - t$ in the last integral shows that it is exactly the same as the first integral. Since the two terms on the left are the same (recalling that $I(x) = I(-x)$), we obtain (3).

Applying equation (3) repeatedly, we find that

$$I(x) = \frac{1}{2}I(x^2) = \frac{1}{2^2}I(x^4) = \cdots = \frac{1}{2^n}I(x^{2^n}).$$

Again we assume that $|x| < 1$, so that $x^{2^n} \to 0$ as $n \to \infty$ and consequently

$$I(x) = \lim_{n\to\infty} \frac{1}{2^n}I(x^{2^n}) = 0.$$

Remark Equation (3) holds for any x. In particular, we have that $I(0) = 0$ and $I(\pm 1) = 0$. The latter equation leads to an added bonus:

$$\int_0^{\pi/2} \ln(\sin\theta)\,d\theta = \int_0^{\pi/2} \ln(\cos\theta)\,d\theta = -\frac{\pi}{2}\ln 2,$$

since, for instance,

$$I(1) = \int_0^{\pi} \ln(2 - 2\cos\theta)\,d\theta = 2\pi\ln 2 + 4\int_0^{\pi/2} \ln(\sin\theta)\,d\theta,$$

and similarly for $I(-1)$. These two integrals are improper. To show convergence, for example, using integration by parts, we have

$$\begin{aligned} \int_0^{\pi/2} \ln(\sin\theta)\,d\theta &= \lim_{\epsilon\to 0} \epsilon\ln(\sin\epsilon) - \lim_{\epsilon\to 0}\int_\epsilon^{\pi/2} \frac{\theta\cos\theta}{\sin\theta}\,d\theta \\ &= -\int_0^{\pi/2} \theta\cot\theta\,d\theta. \end{aligned}$$

Since $\theta\cot\theta$ is Riemann integrable on $[0, \pi/2]$, $\int_0^{\pi/2} \ln(\sin\theta)\,d\theta$ converges.

III. Using parametric differentiation Since $I(x)$ is differentiable for $|x| < 1$, we apply the Leibniz rule to $I(x)$ to find

$$I'(x) = \int_0^{\pi} \frac{-2\cos\theta + 2x}{1 - 2x\cos\theta + x^2}\,d\theta.$$

Clearly, $I'(0) = 0$. We now show that $I'(x) = 0$ for $x \neq 0$. First, we prove that

$$\int_0^{\pi} \frac{1 - x^2}{1 - 2x\cos\theta + x^2}\,d\theta = \pi. \tag{4}$$

The integrand in (4) is called Poisson's kernel, which is used to derive solutions of the two-dimensional Laplace's equation on unit circle [**1**, p.135], and also plays an important role in summation of Fourier series [**2**, pp.163–170]. Computing the value of this integral is often used to show the usefulness of the residue theorem, a relatively advanced tool [**3**, p. 303]. We give a more straightforward method using the half-angle substitution. Setting $t = \tan(\theta/2)$, we have

$$\begin{aligned}\int \frac{1-x^2}{1-2x\cos\theta+x^2}\,d\theta &= 2(1-x^2)\int \frac{dt}{(1-x)^2+(1+x)^2t^2}\\ &= 2\arctan\left(\frac{1+x}{1-x}t\right)+C\\ &= 2\arctan\left(\frac{1+x}{1-x}\tan(\theta/2)\right)+C.\end{aligned}$$

The fundamental theorem of calculus then gives

$$\int_0^{\pi} \frac{1-x^2}{1-2x\cos\theta+x^2}\,d\theta = \lim_{\theta\to\pi} 2\arctan\left(\frac{1+x}{1-x}\tan(\theta/2)\right) = \pi.$$

Using equation (4) and $x \neq 0$, we get

$$I'(x) = \frac{1}{x}\int_0^{\pi}\left(1-\frac{1-x^2}{1-2x\cos\theta+x^2}\right)d\theta = 0.$$

Thus, we have $I'(x) = 0$ for $|\,x\,| < 1$ and so $I(x) =$ constant. Since $I(0) = 0$, we have shown that $I(x) \equiv 0$, for $|\,x\,| < 1$.

IV. Using infinite series We first show that

$$\ln(1-2x\cos\theta+x^2) = -2\sum_{n=1}^{\infty}\frac{x^n}{n}\cos n\theta, \tag{5}$$

where the series converges uniformly for $|\,x\,| < 1$. Once we establish this, integrating (5) with respect to θ from 0 to π, will show that $I(x) = 0$ once again.

To prove (5) and to keep the evaluation at an elementary level, instead of using the Fourier series, we use the relation $2\cos\theta = e^{i\theta} + e^{-i\theta}$ and decompose into partial fractions:

$$\frac{1-x^2}{1-2x\cos\theta+x^2} = \frac{1-x^2}{(1-xe^{i\theta})(1-xe^{-i\theta})} = -1+\frac{1}{1-xe^{i\theta}}+\frac{1}{1-xe^{-i\theta}}.$$

Then the geometric series expansion leads to,

$$\frac{1-x^2}{1-2x\cos\theta+x^2} = 1+2\sum_{n=1}^{\infty}x^n\cos n\theta. \tag{6}$$

The series (6) converges uniformly since $\sum_{n=1}^{\infty}|\,x\,|^n$ converges for $|\,x\,| < 1$. Subtracting 1 from the right-hand side of (6) and then dividing by x, we have

$$\frac{2\cos\theta-2x}{1-2x\cos\theta+x^2} = 2\sum_{n=1}^{\infty}x^{n-1}\cos n\theta. \tag{7}$$

Since the series (7) converges uniformly, integrating from 0 to x term by term, we have established (5) as desired.

Remark As a bonus, we have another proof of integral (4) deduced from series (6).

We have seen a variety of evaluations of the Poisson integral. The interested reader is encouraged to investigate additional approaches.

Acknowledgment. I wish to thank Brian Bradie and the referees for their helpful suggestions.

REFERENCES

1. D. Logan, *Applied Partial Differential Equations*, Springer, New York, 1998.
2. G. Tolstov, *Fourier Series*, Dover Books, New York, 1962.
3. J. E. Marsden & M. J. Hoffman, *Basic Complex Analysis*, Freeman, New York, 1987.

Cycles in the Generalized Fibonacci Sequence modulo a Prime

DOMINIC VELLA
ALFRED VELLA
194 Buckingham Rd.
Bletchley, Milton Keynes, UK, MK3 5JB
Fibonacci@thevellas.com

Since their invention in the thirteenth century, Fibonacci sequences have intrigued mathematicians. As well as modeling the population patterns of overly energetic rabbits, however, they have sparked developments in more serious mathematics. For example, generalized Fibonacci sequences crop up in all manner of situations, from fiber optic networks [**3**] to computer algorithms [**1**] to probability theory [**2**].

In this article, we study generalized Fibonacci sequences $\{G(n)\}$, given by the recurrence relation: $G(n) = aG(n-1) + bG(n-2)$ for $a, b, G(0)$ and $G(1)$ integers. We also study the periods of repetition in such sequences when considered modulo p, a prime. For one particular class of generalized Fibonacci numbers, we find a surprising connection with Fermat's Last Theorem. Other connections between these two seemingly unrelated subjects have been discovered in the past [**8**], but the one unearthed here allows us to calculate the length of these repetitions or *cycles* exactly.

Some useful results When working with the generalized Fibonacci sequences described above, we will need some results to make our lives easier. It is well known that the usual Fibonacci numbers (that is $a = b = 1$, $G(0) = 0$, $G(1) = 1$) can be expressed using Binet's formula [**5**]:

$$\sqrt{5}G(n) = \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

This is easily derived by guessing a solution to the recurrence of the form $G(n) = \lambda^n$, solving for λ, and matching with the initial conditions. In a similar way, many number theory texts (see for example, Niven and Zuckermann [**7**]) prove that the analogous Binet formula for our sequence is

$$(A-B)G(n) = G(1)(A^n - B^n) + G(0)(AB^n - BA^n), \qquad (1)$$