Conversely, suppose d is an odd divisor of n with $d^2 > 2n$, with codivisor d'. Then $d > 2d'$, and if we write $2a + 1 = d$, $k = d'$, then

$$n = (a + 1 - k) + \cdots + a + (a + 1) + \cdots + (a + k)$$

is a partition of n into an even number of consecutive parts. ■

REFERENCE

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Means Generated by an Integral

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For a pair of distinct positive numbers, a and b, a number of different expressions are known as *means*:

1. the arithmetic mean: $A(a, b) = (a + b)/2$
2. the geometric mean: $G(a, b) = \sqrt{ab}$
3. the harmonic mean: $H(a, b) = 2ab/(a + b)$
4. the logarithmic mean: $L(a, b) = (b - a)/(\ln b - \ln a)$
5. the Heronian mean: $N(a, b) = (a + \sqrt{ab} + b)/3$
6. the centroidal mean: $T(a, b) = 2(a^2 + ab + b^2)/3(a + b)$

Recently, Professor Howard Eves [**1**] showed how many of these means occur in geometrical figures. The integral in our title is

$$f(t) = \frac{\int_a^b x^{t+1}\,dx}{\int_a^b x^t\,dx}, \tag{1}$$

which encompasses all these means: particular values of t in (1) give each of the means on our list. Indeed, it is easy to verify that

$$f(-3) = H(a, b), \qquad f\left(-\frac{3}{2}\right) = G(a, b), \qquad f(-1) = L(a, b),$$
$$f\left(-\frac{1}{2}\right) = N(a, b), \qquad f(0) = A(a, b), \qquad f(1) = T(a, b).$$

Moreover, upon showing that $f(t)$ is strictly increasing, we can conclude that

$$H(a, b) \le G(a, b) \le L(a, b) \le N(a, b) \le A(a, b) \le T(a, b), \tag{2}$$

with equality if and only if $a = b$.

To prove that $f(t)$ is strictly increasing for $0 < a < b$, we show that $f'(t) > 0$. By the quotient rule,

$$f'(t) = \frac{\int_a^b x^{t+1} \ln x \, dx \int_a^b x^t \, dx - \int_a^b x^{t+1} \, dx \int_a^b x^t \ln x \, dx}{(\int_a^b x^t \, dx)^2}. \tag{3}$$

Since the bounds of the definite integrals are constant, the numerator of this quotient can be written

$$= \int_a^b x^{t+1} \ln x \, dx \int_a^b y^t \, dy - \int_a^b y^{t+1} \, dx \int_a^b x^t \ln x \, dx$$
$$= \int_a^b \int_a^b x^t y^t \ln x (x - y) \, dx dy.$$

Substituting in a different manner, we write the same numerator as

$$= \int_a^b y^{t+1} \ln y \, dy \int_a^b x^t \, dx - \int_a^b x^{t+1} \, dx \int_a^b y^t \ln y \, dy$$
$$= \int_a^b \int_a^b x^t y^t \ln y (y - x) \, dx \, dy.$$

Averaging the two equivalent expressions shows that this numerator is

$$\frac{1}{2} \int_a^b \int_a^b x^t y^t (x - y)(\ln x - \ln y) \, dx dy > 0,$$

as long as $0 < a < b$. In view of (3), this implies that $f'(t) > 0$. Thus, $f(t)$ is strictly increasing as desired.

We next turn to a refinement of (2). Since

$$f(-2) = \frac{ab(\ln b - \ln a)}{b - a} = \frac{G^2(a, b)}{L(a, b)},$$

the monotonicity of $f(t)$ allows us to deduce the following well-known interpolation inequality:

$$H(a, b) \leq \frac{G^2(a, b)}{L(a, b)} < G(a, b).$$

For more results, some of which have been obtained by other authors [**2**], we define the power mean by

$$M_p(a, b) = \left(\frac{a^p + b^p}{2} \right)^{1/p}.$$

Observing that $M_{1/2}(a, b) = (G(a, b) + A(a, b))/2$ and

$$N(a, b) = \frac{1}{3} \big(G(a, b) + 2A(a, b) \big),$$

we challenge the reader to choose values of t in (1) to show that

$$L(a, b) < M_{1/3}(a, b) < \frac{1}{3} \big(2G(a, b) + A(a, b) \big)$$
$$< M_{1/2}(a, b) < N(a, b) < M_{2/3}(a, b).$$

Following the excellent suggestion of an anonymous referee, for which the author is grateful, we put the discussion in a wider context by generalizing the means defined by (1). We state a set of axioms, which, if satisfied by a class of functions, will entitle those functions to be called means. The axioms will be chosen by abstracting the most important properties of $f(t)$ in (1).

We say that a function $F(a, b)$ *defines a mean* for $a, b > 0$ when

1. $F(a, b)$ is continuous in each variable,
2. $F(a, b)$ is strictly increasing in each variable,
3. $F(a, b) = F(b, a)$,
4. $F(ta, tb) = tF(a, b)$ for all $t > 0$,
5. $a < F(a, b) < b$ for $0 < a < b$.

The reader is invited to show that a necessary and sufficient condition for $F(a, b)$ to define a mean is that for $0 < a \leq b$,

$$F(a, b) = b\, f(a/b),$$

where $f(s)$ is positive, continuous and strictly increasing for $0 < s \leq 1$, and satisfies $s < f(s) \leq 1$, for $0 < s < 1$. In particular, if ϕ is a positive continuous function on $(0, 1]$ and if

$$f(s) = f_\phi(s) = \frac{\int_s^1 x\phi(x)\, dx}{\int_s^1 \phi(x)\, dx},$$

then f satisfies these conditions and

$$F(a, b) = bf(a/b) = \frac{\int_a^b x\phi(x/b)\, dx}{\int_a^b \phi(x/b)\, dx}$$

defines a mean. Moreover, if ψ is positive continuous on $(0, 1]$ and ψ/ϕ is strictly increasing, then $f_\phi < f_\psi$ on $(0, 1)$. This gives a general perspective on the topic of means.

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2. D. S. Mitrinovic, J. E. Pecaric, and A. E. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Boston, 1993, 21–48.

Nonattacking Queens on a Triangle

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Most readers are surely familiar with the problem of placing eight nonattacking queens on a chessboard, and its natural generalization to an $n \times n$ board (see the references at