Condensing a Slowly Convergent Series

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The condensation test for convergence says that, for any monotone decreasing positive sequence $(a_n)_{n\geq 1}$, the convergence of the corresponding series $\Sigma_n a_n$ is equivalent to the convergence of the condensed series $\Sigma_n 2^n a_{2^n}$. This test (sometimes called Cauchy's condensation test) used to be known to every undergraduate, but has lately rather fallen out of fashion. Probably the best account of the condensation test and its numerous generalizations and refinements is still the classic book by Konrad Knopp **[3]**, now republished by Dover. There is also a good treatment in **[2]**.

We can establish that the condensation test works by the following argument. Define $l(1)=l(2)=1$ and $l(n)=k$ for $2^{k-1}<n\leq 2^k$, $k>1$. We can regard $l(n)$ as the integer ceiling of $\log_2 n$. By hypothesis, $a_{2^k}\leq a_n\leq a_{2^{k-1}}$ for all the 2^{k-1} values of n for which $l(n)=k$. This means that

$$a_2+\frac{1}{2}\sum_k 2^k a_{2^k}=2a_2+\sum_k 2^k a_{2^{k+1}}\leq\sum_n a_n\leq a_1+\sum_k 2^k a_{2^k}$$

whence $\Sigma_n a_n$ and $\Sigma_n 2^n a_{2^n}$ converge or diverge together, as asserted.

The test condenses the series by dividing it into chunks, the nth chunk consisting of 2^n terms, and because the terms of the series are decreasing, the test need consider only one representative from each chunk to establish convergence.

Here is an example of the use of the condensation test. If $a_n=1/n$ then $2^n a_{2^n}=2^n\cdot(1/2^n)=1$ for all n. Since we know that $\Sigma_n 1$ diverges, the condensation test tells us that $\Sigma_n 1/n$ diverges. Repeated use of the condensation test tells us that

$$\sum_n\frac{1}{n},\ \sum_n\frac{1}{n\cdot l(n)},\ \sum_n\frac{1}{n\cdot l(n)\cdot l(l(n))},\ \sum_n\frac{1}{n\cdot l(n)\cdot l(l(n))\cdot l(l(l(n)))},\ \text{etc.}$$

all diverge. Similarly, if $a_n=1/n^2$ then $2^n a_{2^n}=2^n\cdot(1/2^n)^2=2^{-n}$. Since we know that $\Sigma_n 2^{-n}$ converges, repeated use of the condensation test tells us that

$$\sum_n\frac{1}{n^2},\ \sum_n\frac{1}{n\cdot[l(n)]^2},\ \sum_n\frac{1}{n\cdot l(n)\cdot[l(l(n))]^2},$$

$$\sum_n\frac{1}{n\cdot l(n)\cdot l(l(n))\cdot[l(l(l(n)))]^2},\ \text{etc.}$$

all converge.

This makes it natural to consider the limiting case $\Sigma_n b_n$ where

$$b_n=\frac{1}{n\cdot\pi(n)}$$

and we define $\pi(1)=1$, $\pi(n)=l(n)\cdot\pi(l(n))$ for $n>1$, so that for example $b_{65536}=1/(65536\cdot 16\cdot 4\cdot 2)$. We assert that the sequence (b_n) shrinks to zero more slowly than do the terms in any of the convergent series given above, but more rapidly than those in any of the corresponding divergent series. For example, it is clear that

$l(n)\cdot l(l(n))/\pi(n) = 1/\pi(l(l(n))) \to 0$ with n, and similarly $l(n)\cdot[l(l(n))]^2/\pi(n) = l(l(n))/\pi(l(l(n))) \to \infty$ with n provided we can show $n/\pi(n) \to \infty$. This can be shown by induction using the fact that $2^n/\pi(2^n) = [2^n/n^2][n/\pi(n)]$.

Given our results so far, it is natural to ask whether $\Sigma_n b_n$ converges or diverges. The condensation test appears to give no help, since $2^n b_{2^n} = 1/\pi(2^n) = 1/(n\cdot\pi(n)) = b_n$, so the series is transformed into itself. But we shall see that by using a refinement of the condensation test we can determine not only that the series converges, but that it does so sufficiently rapidly for us to get a tight estimate of the limit using a geometric series.

To prove convergence of $\Sigma_n b_n$ we argue as follows. Let c_n be the sum of the first 2^n terms, so that $c_n = b_1 + \cdots + b_{2^n}$. It is clear from the definition of b_n that

$$c_n - c_{n-1} = \left[\frac{1}{2^{n-1}+1} + \frac{1}{2^{n+1}+2} + \cdots + \frac{1}{2^n}\right] b_n.$$

Now divide the interval $[1,2]$ into 2^{n-1} equal parts and consider the corresponding upper and lower Riemann sums for $L = \int_1^2 1/x\,dx = \log_e 2 = 0.693147\ldots$. This shows that

$$2^{-n} \geq \int_1^2 \frac{dx}{x} - \left[\frac{1}{2^{n-1}+1} + \frac{1}{2^{n-1}+2} + \cdots + \frac{1}{2^n}\right] \geq 0.$$

Consequently $c_n - c_{n-1} \leq Lb_n$, and summing this over a range of n gives

$$c_{2^n} - c_1 \leq Lb_2 + \cdots + Lb_{2^n} = Lc_n - L \leq Lc_{2^n} - L,$$

whence $c_{2^n} \leq (1.5 - L)/(1-L) \leq 3$ for all n, which establishes convergence of $\Sigma_n b_n$.

To obtain a closer bound on the value of this sum, argue as follows. From the Riemann estimate above we have (dividing through by $n\cdot\pi(n)$)

$$\frac{2^{-n}}{n\cdot\pi(n)} \geq \frac{L}{n\cdot\pi(n)} - [b_{2^{n-1}+1} + b_{2^{n-1}+2} + \cdots + b_{2^n}] \geq 0$$

i.e.,

$$b_{2^n} \geq Lb_n - (c_n - c_{n-1}) \geq 0.$$

Choose n_0, and set $n_1 = 2^{n_0}$, $n_2 = 2^{n_1}$, etc. Summing the previous inequality from $n = n_1 + 1$ to n_2 and noting that $b_{2^{n+1}} \leq \frac{1}{2}b_{2^n}$ gives

$$\tfrac{1}{2}b_{n_2} + \tfrac{1}{4}b_{n_2} + \tfrac{1}{8}b_{n_2} \geq Lb_{n_1+1} + Lb_{n_1+2} + \cdots + Lb_{n_2} + (c_{n_2} - c_{n_1+1}) \geq 0,$$

so we have

$$b_{n_2} \geq L(c_{n_1} - c_{n_0}) - (c_{n_2} - c_{n_1}) \geq 0.$$

Similarly,

$$b_{n_3} \geq L(c_{n_2} - c_{n_1}) - (c_{n_3} - c_{n_2}) \geq 0,$$

$$b_{n_4} \geq L(c_{n_3} - c_{n_2}) - (c_{n_4} - c_{n_3}) \geq 0,$$

and so on. Successively multiplying by L and adding the next inequality gives the following sequence of inequalities

$$b_{n_2} \geq L(c_{n_1} - c_{n_0}) - (c_{n_2} - c_{n_1}) \geq 0,$$

$$b_{n_3} + Lb_{n_2} \geq L^2(c_{n_1} - c_{n_0}) - (c_{n_3} - c_{n_2}) \geq 0,$$

$$b_{n_4} + Lb_{n_3} + L^2 b_{n_2} \geq L^3(c_{n_1} - c_{n_0}) - (c_{n_4} - c_{n_3}) \geq 0,$$

and so on. Summing these and noting that $1 + L + L^2 + \cdots = 1/(1-L)$ gives

$$\frac{b_{n_2} + b_{n_3} + b_{n_4} + \cdots}{1-L} \geq \frac{L}{1-L}(c_{n_1} - c_{n_0}) - \left(\sum_n b_n - c_{n_1}\right) \geq 0,$$

whence

$$\left(1 + \frac{2}{n_3}\right) \cdot \frac{b_{n_2}}{1-L} \geq \frac{c_{n_1} - Lc_{n_0}}{1-L} - \sum_n b_n \geq 0.$$

So, for example, setting $n_0 = 4$ so that $n_2 = 65536$ gives the value of the sum of $\Sigma_n b_n = 2.403448\ldots$ with an accuracy of over six decimal places. The same methods can also be used to evaluate the convergent series introduced earlier, for example

$$\sum_n \frac{1}{n^2} = 1.644934\ldots; \sum_n \frac{1}{n \cdot [l(n)]^2} = 1.910214\ldots;$$

$$\sum_n \frac{1}{n \cdot l(n) \cdot [l(l(n))]^2} = 2.068641\ldots$$

If further places are required for $\Sigma_n b_n$ (so that we need accurate values for c_{32}, c_{64}, etc.) then we can estimate $c_{n+1} - c_n$ to within 2^{-3n} as follows. Recall that

$$c_n - c_{n-1} = \left[\frac{1}{2^{n-1}+1} + \frac{1}{2^{n-1}+2} + \cdots + \frac{1}{2^n}\right] b_n.$$

By using Simpson's rule

$$\begin{aligned} 3L &= \left[\frac{1}{2^{n-1}} + \frac{4}{2^{n-1}+1} + \frac{2}{2^{n-1}+2} + \frac{4}{2^{n-1}+3} + \cdots + \frac{4}{2^n - 1} + \frac{1}{2^n}\right] \\ &= \frac{1}{2^{n-1}} + 4\left[\frac{1}{2^{n-1}+1} + \frac{1}{2^{n-1}+2} + \cdots + \frac{1}{2^n}\right] \\ &\quad - \left[\frac{1}{2^{n-2}+1} + \frac{1}{2^{n-2}+2} + \cdots + \frac{1}{2^{n-1}}\right] - \frac{1}{2^n} \end{aligned}$$

to within order 2^{-3n}, so we derive the recurrence relation

$$c_n - c_{n-1} = \frac{b_n}{4}\left[\frac{c_{n-1} - c_{n-2}}{b_{n-1}} + 3L - 2^{-n}\right]$$

to the order $2^{-3n}n^{-1}$. For example, if we require the value of $\Sigma_n b_n$ to 12 decimal places, we can calculate c_{13}, c_{14} directly, then use the recurrence relations to calculate $c_{32} = 0.000722028313\ldots$, then use the previous estimate with $n_0 = 5$ to give the value of the sum to the required accuracy.

It is worth noting that partitioning the series $\Sigma 1/n^2$ into chunks according to the rule $n_1 = 2n_0$, $n_2 = 2n_1$, etc. and summing each of the individual chunks also produces an (approximately) geometrically decreasing sequence of chunk sums. The condensation test corresponds to exponential increase in chunk length. The series Σb_n considered here requires more drastic (i.e. super exponential) growth of chunk lengths in order to obtain a sustainable geometric rate of decay for the sequence of chunk sums. This is a slightly counter-intuitive form of the general observation that the more slowly convergent the series, the higher the order of condensation required to reduce it to the geometric case, and is the basis of some of the results in the analysis of Tsierelsen spaces (see [1]).

The fact that $2 < e$ (so that $L < 1$) is vital to convergence. If we define $l_a(n)$ to be the integer ceiling of $\log_a n$ and $\pi_a(n) = l_a(n) \cdot \pi_a(l_a(n))$ then methods similar to those used earlier show that $\Sigma_n [n \cdot \pi_a(n)]^{-1}$ converges if, and only if, $a < e$. The gentle reader is invited to set $a = e$ and modify the denominator so as to find a series that requires an even higher order of condensation to establish convergence.

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On Characterizations of the Gamma Function

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1. Introduction It is well known that the gamma function $\Gamma(x) > 0$ on $(0, \infty)$ satisfies the functional equation $\Gamma(x+1) = x\Gamma(x)$ and the initial condition $\Gamma(1) = 1$. However, these two properties do not characterize the gamma function. Rather surprisingly, the additional assumption of the convexity of $\log \Gamma(x)$ is sufficient for a characterization, a fact discovered by Bohr and Mollerup [**1**]. For a proof, see Artin's book [**4**, **5**] or Rudin's book [**6**], or the last section of this paper. Note that the initial condition in the characterization is not essential, for if f is a positive function on $(0, \infty)$ such that $f(x+1) = xf(x)$ then $g(x) = f(1)^{-1}f(x)$ is a positive function that satisfies the same functional equation and $g(1) = 1$.

A second characterization formulated and proved by Laugwitz and Rodewald [**2**] says that the convexity of $\log \Gamma(x)$ can be replaced by the property, call it property (L), that the function $L(x) = \log \Gamma(x+1)$ satisfies

$$L(n+x) = L(n) + x\log(n+1) + r_n(x), \quad \text{where } r_n(x) \to 0 \text{ as } n \to \infty. \tag{L}$$

However, they did not show how this property is related to the convexity of $\log \Gamma(x)$. The original idea of the second characterization goes back to Euler [**3**].

In the present paper we give a third characterization of the gamma function and then show how these three characterizations are related.

2. A third characterization In property (L), the use of logarithms is not essential and without logarithms the expression on the right-hand side becomes a product instead of a sum. We might therefore expect that a modified property (L) will give us a characterization that is closer to the product expression of the gamma function. With this in mind, we modify property (L) as follows: The gamma function satisfies the following property

$$\Gamma(x+n) = \Gamma(n)n^x t_n(x), \quad \text{where } t_n(x) \to 1 \text{ as } n \to \infty.$$