Limitless Integrals and a New Definition of the Logarithm

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The integrals we consider are

$$\int_a^b x^n \, dx \quad \text{for integers } n \neq -1. \tag{1}$$

We show by a clever choice of the evaluation points in the partition intervals used to define the Riemann sums for these integrals that we can arrange for these sums to agree exactly with the value of the integral. Thus the integrals can be evaluated without having to take limits. We also show how the inequalities used here lead to a new definition of the natural logarithm that does not involve limits, series or integrals, or even the exponential function.

A Riemann sum for the integral in (1) is

$$\sum_{i=1}^{m} u_i^n (x_i - x_{i-1}) \tag{2}$$

where $(x_0, x_1, x_2, \dots x_m)$ is an ordered partition of $[a, b]$ and $x_{i-1} < u_i < x_i$ for $i = 1, 2, \dots m$. For convenience we suppose that $0 \leq a < b$, and that $a > 0$ when n is negative. Since the case $n = 0$ is trivial, we assume that $n \neq 0, -1$. Let $u_i = M_n(x_i, x_{i-1})$, where $M_n(u, v)$ is defined for $u, v > 0$ by

$$M_n(u, u) = u \tag{3}$$

and

$$M_n(u, v) = \left[\frac{u^{n+1} - v^{n+1}}{(n+1)(u - v)} \right]^{1/n}$$

for $u \neq v$. If $0 \leq v < u$ then, as we show below, $v < M_n(u, v) < u$ and so $x_{i-1} < u_i < x_i$. With this choice of u_i we have

$$u_i^n (x_i - x_{i-1}) = \left(x_i^{n+1} - x_{i-1}^{n+1} \right) / (n+1)$$

and so the Riemann sum (2) telescopes and has the value

$$\left(b^{n+1} - a^{n+1} \right) / (n+1). \tag{4}$$

The mesh size $\max_i (x_i - x_{i-1})$ may be made arbitrarily small, so, invoking the usual theorems on the existence of the integral (see **[1]**, for example) we see that the value of the integral (1) must be the expression (4). For n negative we only need these results for $v > 0$.

What makes the method work is the fact that each expression M_n defined by (3) is an average. We note in fact that $M_1(u, v) = (u + v)/2$ and $M_{-2}(u, v) = (uv)^{1/2}$ are the arithmetic and geometric means respectively. For n a positive integer the mean-value property of M_n follows from the formula

$$M_n(u, v) = \left[(u^n + u^{n-1} v + \cdots + v^n) / (n+1) \right]^{1/n} \tag{5}$$

and the fact that the radicand is the arithmetic average of $n+1$ non-negative numbers, all not equal, the largest of which is u^n and the smallest of which is v^n. Since we have ruled out $n=0$ or -1, and $n=-2$ is the easy case of the geometric mean, we only have to consider $n \leq -3$ among the negative integers. Some easy algebra shows that the inequality $M_{-n}(u,v) > v$ is equivalent to the inequality

$$M_{n-2}\left(\frac{v}{u},1\right) < \left(\frac{u}{v}\right)^{1/(n-2)},$$

and this follows from $u/v > 1$ and the result already established that M_{n-2} is an average. The inequality $M_n(u,v) < u$ for n negative is proved similarly. Various generalizations of the M_n are discussed in **[2]**.

When $n=-1$ in (1) this method seems incapable of giving an evaluation of the integral (1) that does not depend on dealing with limits because of a lack of a suitable elementary definition of the logarithm. Below we make it work by giving a new definition of the logarithm. We note first that the usual tactic in this situation is to define the logarithm by setting

$$L(x) = \int_1^x \frac{1}{t}\,dt. \tag{6}$$

The functional equation of the logarithm, $L(xy) = L(x) + L(y)$, is then obtained from this definition (see **[3]**, p. 93, problems 3 and 4), and this functional equation is easily transformed into Cauchy's functional equation, $f(x+y) = f(x) + f(y)$, by introducing $f(x) = L(a^x)$, $a > 1$. From (6) the continuity of L follows as does the fact that the monotone function L takes on every real value exactly once. From the form of the continuous solutions of Cauchy's functional equation ($f(x) = cx$) it follows that $L(x) = \log_e x$, where e is defined by $L(e) = 1$.

We suggest another method of attack: We note that L defined by (6) must satisfy the inequality

$$u^{-1} < \frac{L(u) - L(v)}{u - v} < v^{-1} \tag{7}$$

for all positive u and v, $v < u$. We assert that this inequality defines the logarithm up to an additive constant. With the added condition $L(1) = 0$, we make this our definition of the logarithm. The unique solution of (7) is then $L(x) = \log_e x$ (with e now defined as usual).

We give two proofs of this. Our first proof depends on the calculus and runs as follows (we omit the details): We show in succession that any solution of (7) is necessarily strictly increasing, continuous, differentiable, and satisfies $L'(x) = x^{-1}$. The assertion now follows. For a similar elementary definition of the exponential function, see **[4]**, part (a).

The following treatment of (7) is more in the spirit of this paper and depends only on the existence of the integral. Let $n=-1$ in (1) and (2) and let u_i in the Riemann sum be defined by

$$u_i^{-1} = \frac{L(x_i) - L(x_{i-1})}{x_i - x_{i-1}},$$

where L is any solution of (7). By (7) $x_{i-1} < u_i < x_i$, and the value of the Riemann sum is $L(b) - L(a)$. It follows as before that

$$L(b) - L(a) = \int_a^b \frac{1}{t}\,dt.$$

Thus any solution of (7) together with $L(1)=0$ is necessarily given by (6).

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A Note on Cauchy Sequences

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In undergraduate courses we teach that a sequence of real numbers converges if, and only if, it is a Cauchy sequence. Usually, the students have no problems with the necessary condition. Our aim in this note is to clarify the sufficient condition with the use of pairs of monotone sequences. The notation is standard; we follow **[1]**.

First of all, we recall the notion of pairs of monotone sequences:

Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be sequences of real numbers. We say that $(\{a_n\}_{n\in\mathbb{N}},\{b_n\}_{n\in\mathbb{N}})$ is a *pair of monotone sequences* if:

(1) $\{a_n\}_{n\in\mathbb{N}}$ is nondecreasing and $\{b_n\}_{n\in\mathbb{N}}$ is nonincreasing, and
(2) $a_n\le b_n$ $\forall n\in\mathbb{N}$.

Plainly, $a_p\leqslant b_q$, for all $p,q\in\mathbb{N}$ so there exist $a=\sup\{a_n\colon n\in\mathbb{N}\}\in\mathbb{R}$ and $b=\inf\{b_n\colon n\in\mathbb{N}\}\in\mathbb{R}$, and $a\le b$.

When can we assure that $a=b$? If the following condition is fulfilled:

$$\forall\varepsilon>0\ \exists k\in\mathbb{N}\quad\text{such that } b_k-a_k<\varepsilon,\qquad a=b. \tag{$*$}$$

We recall the definition of *limit superior* and *limit inferior* of a bounded sequence of real numbers. Let $\{x_n\}_{n\in\mathbb{N}}\subset\mathbb{R}$ be a bounded sequence. For each $n\in\mathbb{N}$ define

$$y_k=\inf\{x_n\colon n\ge k\}\quad\text{and}\quad z_k=\sup\{x_n\colon n\ge k\}.$$

Let $y=\lim_{k\to\infty}y_k=\sup\{y_k\colon k\in\mathbb{N}\}$ and $z=\lim_{k\to\infty}z_k=\inf\{z_k\colon k\in\mathbb{N}\}$. The numbers $y,z\in\mathbb{R}$ are called the *limit inferior* of $\{x_n\}_{n\in\mathbb{N}}$ and the *limit superior* of $\{x_n\}_{n\in\mathbb{N}}$ respectively. (Note that $(\{y_n\}_{n\in\mathbb{N}},\{z_n\}_{n\in\mathbb{N}})$ is a pair of monotone sequences). Furthermore, we recall that every Cauchy sequence is bounded and a sequence of real numbers is convergent if, and only if, the limits superior and inferior exist in $\mathbb{R}$ and are equal.

With these remarks, we are ready. Suppose that $\{x_n\}_{n\in\mathbb{N}}\subset\mathbb{R}$ is a Cauchy sequence. Since $\{x_n\}_{n\in\mathbb{N}}$ is bounded, $y=\liminf x_n\in\mathbb{R}$, $z=\limsup x_n\in\mathbb{R}$. To show $\{x_n\}_{n\in\mathbb{N}}$ converges, we only have to prove that $y=z$.

We know that

$$\forall\varepsilon>0\ \exists k\in\mathbb{N}\quad\text{such that }\forall m,n\ge k,\qquad |x_m-x_n|<\varepsilon.$$

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