2. Which rectangular boards with k squares removed can be tiled with trominoes?

Proposition 3 showed that a $(2i)\times(3j)$ (nondeficient) board can be tiled with trominoes. We can ask:

3. Which (nondeficient) rectangular boards can be tiled with trominoes?

A new set of problems results if we ask the preceding questions about some other kind of polyomino. In this connection, de Bruijn [**1**] proved that if an $n\times m$ board is tiled by $a\times b$ rectangular polyominoes, then either a divides n or a divides m. Actually, de Bruijn's result was valid in an arbitrary number of dimensions; we have stated only the two-dimensional case. Of course, all of the above questions can be posed in an arbitrary number of dimensions.

Finally, once we have a tiling of a board, we can ask:

4. How many tilings of a particular type are there?

References

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Three Aspects of Fubini's Theorem

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Which of the three propositions in the box—(1), (2) or (3)—would you consider to be the most palpably true? Our first choice is (1), while (3) is second, and (2) is a close third. This is because

Let $f(x,y)$, $\frac{\partial}{\partial x}g(x,y)$, and $\frac{\partial^2}{\partial y\partial x}h(x,y)$ be continuous real-valued functions in the rectangle $\{(x,y)\colon a\leqslant x\leqslant b, c\leqslant y\leqslant d\}$. Then:

$$(1)\quad \int_a^x\int_c^y f(u,v)\,dv\,du=\int_c^y\int_a^x f(u,v)\,du\,dv,$$

$$(2)\quad \frac{\partial}{\partial x}\int_c^y g(x,v)\,dv=\int_c^y\frac{\partial}{\partial x}g(x,v)\,dv,$$

$$(3)\quad \frac{\partial^2}{\partial y\partial x}h(x,y)=\frac{\partial^2}{\partial x\partial y}h(x,y).$$

the geometrical evidence for (1) provides a more compelling argument than the naturalness and sense of order of (2) and (3). In fact, (3)'s interpretation using velocities actually *detracts* from its believability (as we shall see)!

These statements are surprising in light of the fact that *using only the fundamental theorem of* *calculus and some routine manipulations, any one of these propositions can be derived from any other.*

Many of our observations can be found in [**2**], and some of the ideas are suggested by exercises in [**1**, p. 793], [**3**, p. 61], and [**4**, pp. 464–465]. Nevertheless, they are missing from contemporary calculus texts and deserve occasional airings. In addition to bringing [**2**] back to light, our goal here is to emphasize the intuitive content of this circle of ideas.

Statement (1), a special case of Fubini's theorem, can be interpreted as follows:

> One gets just as much tomato to eat if he slices it from left to right or from back to front.

Compare this with the mental gymnastics required to untangle the interpretation of (3):

> A person walks on a hillside and points a flashlight along a tangent to the hill; then the rate at which the beam's direction changes when walking south and pointing east equals its rate of change when walking east and pointing south.

We leave the interpretation of (2) to the reader. (Hint: The left side of (2) is the rate of change of the cross-sectional area of the tomato slices mentioned above. Does your interpretation of (2) convince you of its validity?)

Proofs that the statement (i) implies ($i \pm 1$) are readily found in textbooks (or see [**2**]). As a typical example, here is the standard proof that (1) implies (2). We assume (1) and define $f(x, y) = \frac{\partial}{\partial x} g(x, y)$. That is,

$$\int_a^x f(u, y)\, du = g(x, y) - g(a, y).$$

Then

$$\begin{aligned}\frac{\partial}{\partial x}\int_c^y g(x, v)\, dv &= \frac{\partial}{\partial x}\int_c^y\left(\int_a^x f(u, v)\, du + g(a, v)\right) dv\\ &= \frac{\partial}{\partial x}\int_c^y\int_a^x f(u, v)\, du\, dv + \frac{\partial}{\partial x}\int_c^y g(a, v)\, dv.\end{aligned}$$

Since $\int_c^y g(a, v)\, dv$ is a function of y only, its partial derivative with respect to x is zero, and (having assumed (1))

$$\begin{aligned}\frac{\partial}{\partial x}\int_c^y g(x, v)\, dv &= \frac{\partial}{\partial x}\int_a^x\int_c^y f(u, v)\, dv\, du\\ &= \int_c^y f(x, v)\, dv\\ &= \int_c^y \frac{\partial}{\partial x} g(x, v)\, dv.\end{aligned}$$

The proof's only nontrivial steps use the fundamental theorem of calculus. Indeed, one rather undesirable feature of this proof is that the details make it seem as if something more is involved. Let us therefore change our notation to one of operators to bring out the essence of the above argument. Define

$$D_x f := \frac{\partial f}{\partial x}, \qquad D_x^{-1} f := \int_a^x f(u, y)\, du,$$

$$D_y f := \frac{\partial f}{\partial y}, \quad \text{and} \quad D_y^{-1} f := \int_c^y f(x, v)\, dv.$$

In this notation, statements (1), (2), (3) become

$$D_x^{-1}D_y^{-1}=D_y^{-1}D_x^{-1}, \tag{1}$$

$$D_xD_y^{-1}=D_y^{-1}D_x, \tag{2}$$

and

$$D_xD_y=D_yD_x. \tag{3}$$

The fundamental theorem of calculus for $f=f(z)$ is *essentially* $D_zD_z^{-1}f=D_z^{-1}D_zf=f$, where "essentially" means that $D_z^{-1}D_zf$ should have a constant of integration. Of course, in the present context that constant eventually disappears (much as it did in the detailed proof), a fact that can conveniently be left as an exercise. With this warning, the proof that (1) implies (2) now reads

$$D_xD_y^{-1}\underset{\text{F.T.}}{=}D_xD_y^{-1}\left(D_x^{-1}D_x\right)\underset{(1)}{=}D_x\left(D_x^{-1}D_y^{-1}\right)D_x\underset{\text{F.T.}}{=}D_y^{-1}D_x$$

Here is (2) implies (3):

$$D_yD_x\underset{\text{F.T.}}{=}D_yD_xD_y^{-1}D_y\underset{(2)}{=}D_yD_y^{-1}D_xD_y\underset{\text{F.T.}}{=}D_xD_y.$$

The proofs that (3) implies (2) and (2) implies (1) can be obtained by interchanging D with D^{-1} in the lines above.

We should emphasize that because $D^{-1}Df$ differs from f by a constant, the above argument does not constitute a rigorous proof that (i) implies (i − 1). It is, however, an amusing exercise to decode such a symbolic argument to check that each constant of integration really does disappear. Here, for example, is a proof that (3) implies (2) (by decoding $D_y^{-1}D_x=D_y^{-1}D_xD_yD_y^{-1}=D_y^{-1}D_yD_xD_y^{-1}=D_xD_y^{-1}$):

$$\begin{aligned}\int_c^y\frac{\partial}{\partial x}g(x,v)\,dv&\underset{\text{F.T.}}{=}\int_c^y\frac{\partial}{\partial x}\left(\frac{\partial}{\partial v}\int_c^v g(x,t)\,dt\right)dv\\&\underset{(3)}{=}\int_c^y\frac{\partial}{\partial v}\frac{\partial}{\partial x}\int_c^v g(x,t)\,dt\,dv\\&\underset{\text{F.T.}}{=}\frac{\partial}{\partial x}\int_c^y g(x,t)\,dt-\frac{\partial}{\partial x}\int_c^c g(x,t)\,dt\\&=\frac{\partial}{\partial x}\int_c^y g(x,v)\,dv.\end{aligned}$$

The ideas touched upon in this note seem to be appropriate for any calculus course, rigorous or not. At one level they provide an attractive way of proving (3): merely explain how it follows quickly from (1). At any level they provide the opportunity to stress normally unseen connections while providing one more chance to show (and show off) the power of the fundamental theorem of calculus.

Note finally that one can easily avoid the intermediate proposition (2), since (3) follows *directly* from (1):

$$D_yD_x=D_yD_x\left(D_y^{-1}D_x^{-1}\right)D_xD_y=D_yD_x\left(D_x^{-1}D_y^{-1}\right)D_xD_y=D_xD_y.$$

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