The Power Rule and the Binomial Formula

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Perhaps the most common proof [C. H. Edwards, Jr. and David E. Penney, *Calculus and Analytic Geometry*, Prentice Hall, Englewood Cliffs, NJ, 1982, p. 67] of the power rule,

$$\frac{d}{dx}[x^n] = nx^{n-1},$$

$n = 0, 1, \ldots$ relies on the binomial formula:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

A number of texts [M. A. Munem and D. J. Foulis, *Calculus with Analytic Geometry*, 2nd Ed., Worth, NY, 1984, p. 106], however, first prove the product rule and then use a simple induction argument to establish the power rule. The purpose of this note is to reverse the usual order, that is, use the power rule to prove the binomial theorem. The most common proof of the binomial theorem uses a counting argument that is not always convincing to students. The following argument can be thought of as a simple application of the calculus.

Simple induction shows that if f is a polynomial of degree n, then

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k.$$

Now if $g(x) = (x+b)^n$, then $g(0) = b^n$ and the power rule yields $g'(0) = nb^{n-1}$, $g''(0) = n(n-1)b^{n-2}$, and, in general,

$$g^{(k)}(0) = k!\binom{n}{k} b^{n-k}.$$

Therefore,

$$(x+b)^n = g(x) = \sum_{k=0}^{n} \frac{g^{(k)}(0)}{k!} x^k = \sum_{k=0}^{n} \binom{n}{k} b^{n-k} x^k.$$

The binomial theorem now follows by letting $x = a$.

From the Steps of Olympus

...the Institute regulars made allowances and exceptions for von Newmann that they would not have considered for anyone else. For all his dirty-handed messing with computers, he was still one of the high-minded luminaries, one of the immortals, one of the gods that trod upon Earth. "The story used to be told about him in Princeton," Herman Goldstine has written, "that while he was indeed a demi-god, he had made a detailed study of humans and could imitate them perfectly."

Ed Regis, *Who got Einsteins' Office? Eccentricity and Genius at the Institute for Advanced Study*, Addison-Wesley