## Classifying Row-reduced Echelon Matrices

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One of the fundamental theorems of linear algebra states that the row-reduced echelon form of a *given* $m \times n$ matrix is unique. Yet, there are many row-reduced echelon forms associated with the set of *all* $m \times n$ matrices, for given $m$ and $n$ with $n > 1$. We shall characterize them and count their number.

Recall that a matrix is said to be in *row-reduced echelon form* if the following conditions are all satisfied:

(i) In each row that does not consist entirely of zeros, the first nonzero entry is a one (known as a *leading one*).
(ii) In each column that contains a leading one of some row, all other entries are zero.
(iii) In any two rows with nonzero entries, the leading one of the higher row is farther to the left.
(iv) Any row that contains only zeros is lower than all rows that have some nonzero entries.

In any row-reduced echelon matrix, we shall refer to the (positioned) zeros required by conditions (i), (ii), and (iv) of the definition as *forced zeros*.

As an example, consider all possible $2 \times 3$ row-reduced echelon matrices. Each of these can be written as a special case of one of the following, where $x$ and $y$ are arbitrary.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & x & y \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & x \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \end{bmatrix}, \quad \begin{bmatrix} 1 & x & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The displayed ones and zeros are *leading* ones and *forced* zeros. We shall refer to entries that are neither leading ones nor forced zeros as *undetermined* entries. Thus, we have three *types* of entries for each matrix.

Although there are infinitely many matrices represented above, there are only seven different "classes" of them. To make this idea precise, we shall say that two row-reduced echelon matrices are *type-equivalent* if each pair of corresponding entries is of the same type. For example, the matrices

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 8 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

are type-equivalent, but the matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are not. In the latter case, the entries in the following positions are of different types: $(1,2)$, $(1,3)$, $(2,2)$ and $(2,3)$.

The relation induced by "type-equivalence" is an equivalence relation—it is symmetric, reflexive and transitive. It therefore partitions the set of all $m \times n$ row-reduced echelon matrices into equivalence classes. The above pair of $3 \times 4$ matrices belongs to the same equivalence class, whereas the above pair of $2 \times 3$ matrices does not. In this context, the seven $2 \times 3$ row-reduced echelon matrices displayed earlier are the seven equivalence classes that partition the set of all $2 \times 3$

row-reduced echelon matrices. Note that

$$7=\binom{3}{0}+\binom{3}{1}+\binom{3}{2}.$$

In general, the number of distinct type-equivalent $m \times n$ matrices (that is, the number of equivalence classes induced by the "type" relation) is given by

$$N(n,m)=\sum_{k=0}^{\min(m,n)}\binom{n}{k}.$$

To verify this, first observe that the positions of the leading ones in any row-reduced echelon matrix determine the type of *all* the entries in that matrix. (They determine the positions of the forced zeros, and hence those of the undetermined entries.) Thus, all we need show is that the number of ways that the leading ones can be arranged is given by the formula above. Our approach will be to do this for matrices of *rank* $k$ (that is, with $k$ nonzero rows in their reduced form) and then sum the results from rank 0 to rank $\min(m,n)$ (the largest possible).

Suppose $A$ is an $m \times n$ row-reduced echelon matrix of rank $k$. Then $A$ has exactly $k$ leading ones. These leading ones, located in the first $k$ rows of $A$, must occur in $k$ distinct columns. Once the columns are specified, the positions of the leading ones are completely determined since they form "stair steps" down to the right. Since there are $\binom{n}{k}$ ways of choosing $k$ objects from a collection of $n$ distinct objects, there are $\binom{n}{k}$ ways of positioning the leading ones. Thus, there are exactly $\binom{n}{k}$ equivalence classes for $m \times n$ row-reduced echelon matrices of rank $k$. Summing over $k=0,1,\ldots,\min(m,n)$ completes the proof.

As an example, observe that the number of distinct type-equivalent $3 \times 3$ row-reduced echelon matrices is

$$N(3,3)=\binom{3}{0}+\binom{3}{1}+\binom{3}{2}+\binom{3}{3}=8.$$

It is no coincidence that the answer turned out to be $2^3$. Indeed, the number of distinct type-equivalent square matrices of order $n$ is equal to $2^n$. This follows immediately from

$$N(n,n)=\sum_{k=0}^{n}\binom{n}{k},$$

since $(1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^k$ yields $2^n=\sum_{k=0}^{n}\binom{n}{k}$.

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## Using Riemann Sums in Evaluating a Familiar Limit

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In "Alternate Approaches to Two Familiar Results" [CMJ 15 (November 1984) 422–426], Norman Schaumberger presented an elementary proof that

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e}. \tag{1}$$

His proof stimulated our thinking and this led to the following geometrically motivated proofs of (1), based on approximating $\int_0^1 \ln x\,dx$ by Riemann sums.