Countability via Bases Other Than 10

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Certain results concerning the infinite seem as paradoxical today as they did in the seventeenth century. The fact that some sets possess proper subsets to which they are equivalent is as bewildering to today's undergraduates as it was to Galileo some three hundred years ago. However, taking that strange property concerning proper subsets to be the definition of an infinite set allows us to begin a systematic investigation of the infinite. We deem two sets *equivalent* if and only if there exists a one-to-one correspondence from one set to a subset of the other and vice versa. We then say that *the cardinalities of the sets are equal.* This method of measuring the infinite produces some strikingly counterintuitive results.

In this short note, I will present alternative proofs that two well-known sets have the same cardinality as the set $\mathbb{N}$ of natural numbers. Such sets are said to be *countable* (or denumerable). The ideas underlying these proofs, while not new [**2**], are also not the ones found in most introductory texts. Additionally, these alternative proofs seem much more accessible to the computer literate mathematics student of today than those standard proofs to which we have grown accustomed.

Students sometimes find it hard to believe that the set of rational numbers is countable. This "proof without words," due to Cantor (see [**3**]), embodies the standard method of constructing a one-to-one map from the set of $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$:

$$\begin{array}{ccccc}
\frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \cdots \\
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \cdots \\
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \cdots \\
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}$$

But it has its drawbacks; additional adjustments must be made both to account for the fact that rational numbers are not simply fractions but *equivalence classes* of fractions, and to include the negative rationals. I now offer an alternative proof of the denumerability of the rationals that neatly avoids these complications.

First recall that every natural number has a unique base 12 representation: a finite string of symbols, with a nonzero leading entry, chosen from $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, -, /\}$, where $-$ represents 10 and $/$ represents 11. The representation is unique. For example,

$$3 - / = 3(12^2) + 10(12^1) + 11(12^0) = 563.$$

This base 12 representation of the natural numbers will allow us to construct a one-to-one correspondence from the set of rational numbers, $\mathbb{Q}$, to a subset of the natural numbers, $\mathbb{N}$. (A one-to-one correspondence from $\mathbb{N}$ into $\mathbb{Q}$ is the obvious "inclusion.") The mapping from $\mathbb{Q}$ into $\mathbb{N}$ is defined by interpreting the symbols of

the reduced form of a rational number such as $-23/17$ as the base 12 representation of a natural number. Thus,

$$-23/17 \mapsto 2536579,$$

since

$$10(12^5) + 2(12^4) + 3(12^3) + 11(12^2) + 1(12^1) + 7(12^0) = 2536579.$$

By convention, negative rationals are written with the negative sign in the numerator and the reduced form of zero is $-0/1$, so the leading symbol is nonzero. The uniqueness of the reduced form of rational numbers and the uniqueness of the base 12 representation of the natural numbers imply that our map is well defined and one-to-one.

Exercise 1. Describe the range.

This method of proof can also be adapted to prove that the set of algebraic numbers is countable. Recall that an algebraic number is a number satisfying a polynomial equation over $\mathbb{Z}$. Cantor's proof that the algebraic numbers are countable involves an argument based on the concept of "height" of a polynomial. The proof may be found in [**3**]. A simpler proof based on the degree of a polynomial along with a diagonal counting argument may be found in [**1**]. We will prove the denumerability via a base 14 argument. We begin by showing that the set $\mathbb{Z}[x]$ of polynomials over $\mathbb{Z}$ is countable. Consider the base 14 representation of the natural numbers as finite strings of symbols, with a nonzero leading entry, chosen from $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, x, +, -, \hat{}\}$, where x represents 10, $+$ represents 11, and so on. This representation allows us to construct a one-to-one correspondence from $\mathbb{Z}[x]$ to a subset of $\mathbb{N}$. The mapping is best described by example:

$$\begin{aligned} 3x\hat{}2 - 2x + 4 \longmapsto\ & 3(14^8) + 10(14^7) + 13(14^6) + 2(14^5) + 12(14^4) \\ & + 2(14^3) + 10(14^2) + 11(14^1) + 4(14^0) = 5580930422. \end{aligned}$$

Again, by convention, polynomials are written with powers in descending order, the leading coefficient never includes the $+$ sign, and the zero polynomial is written as -0. Hence each polynomial uniquely defines a natural number. Since $\mathbb{N}$ is obviously embedded in $\mathbb{Z}[x]$, we see that the set of polynomials over $\mathbb{Z}$ is countable. Also, because each polynomial has only a finite number of roots, we may invoke the fact that a countable union of finite sets is countable and conclude that the set of algebraic numbers is countable.

Exercise 2. Use an appropriate base 11 representation of the natural numbers to show that the set of integers is countable.

Exercise 3. Define the word "sentence." Show that the set of all "sentences" is denumerable.

Exercise 4. Is the set of all ideas a countable set? Explain.

References

1. G. Birkhoff and S. MacLane, *A Survey of Modern Algebra*, MacMillan, New York, 1941.
2. S. L. Campbell, Countability of sets, *American Mathematical Monthly* 93 (1986) 480–481.
3. H. Eves, *An Introduction to the History of Mathematics*, 6th ed., Saunders, New York, 1990.

Mercator's Rhumb Lines: A Multivariable Application of Arc Length

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One of the areas in calculus that suffer from a lack of approachable application problems is arc length. The following historical situation can be understood by any college student and solved by any third-semester calculus student. Our main focus will be an arc length problem in $\mathbb{R}^3$ with a closed-form solution.

In 1569, Gerardus Mercator (1512–1594) created a new map that changed the world, both literally and figuratively. Until then, as sailors navigated the open ocean by following a fixed compass bearing, they could not map a straight line from point A to point B that would correspond to a path of constant compass bearing on the earth's surface (except in a few special cases).

Mercator's accomplishment allowed navigators to chart paths of constant bearing, *rhumb lines*, between any two points on the map. Using the Mercator map, the navigator could now draw a line from A to B, then measure the angle this line makes with a line of longitude. The ship would arrive at destination B by sailing the measured bearing for the entire trip. Edward Wright, an English mathematician of the time, gave an excellent method for visualizing Mercator's map [**5**]:

> Suppose the spherical earth to be represented by a balloon covered with a network of parallels of latitude and meridians equally spaced. Let the balloon be placed inside a cylinder whose inside diameter is such that the equator of the sphere just touches the walls of the cylinder. Then let the balloon be inflated. As it expands, the curved meridians become straightened and flattened against the walls of the cylinder. At the same time, each successive parallel finally comes to rest against the walls of the cylinder. This process goes on to infinity, because the polar regions and the poles themselves can never be pressed against the walls. If the balloon remains against the sides of the cylinder, and the cylinder is unrolled and flattened, the impress is a Mercator projection of the World.

Of course, as is the case with many technological innovations, it took a long time for Mercator's map to gain widespread acceptance. In 1581, Michiel Coignet of Antwerp objected to Mercator's chart [**1**]:

> There was no sense in laying off a course on a chart according to compass direction as it appeared on the chart. The rhumb lines radiating from the compass rose might be straight lines on the chart but the same rhumbs applied to the spherical surface of the ocean would produce a series of spiral curves that would take a navigator precisely nowhere.

Four hundred years later the questions remain: "Does a rhumb line lead anywhere?" and "If so, how long is the path?"