This can be rewritten as

$$r^2(s-b+s-a)=(s-a)(s-b)(s-c)-r^2(s-c). \tag{7}$$

In like manner,

$$\frac{\alpha}{2}=\frac{\pi}{2}-\left(\frac{\beta}{2}+\frac{\gamma}{2}\right) \quad \text{and} \quad \frac{\beta}{2}=\frac{\pi}{2}-\left(\frac{\alpha}{2}+\frac{\gamma}{2}\right)$$

yield

$$r^2(s-c+s-b)=(s-a)(s-b)(s-c)-r^2(s-a) \tag{8}$$

and

$$r^2(s-a+s-c)=(s-a)(s-b)(s-c)-r^2(s-b), \tag{9}$$

respectively. Adding (7), (8), (9), and recalling that $2s=a+b+c$, we find

$$r^2=\frac{(s-a)(s-b)(s-c)}{s}. \tag{10}$$

Thus, substitution of (10) into (2) yields (1).

Editor's Note: There are numerous proofs of Heron's formula which do not refer to an inscribing circle. In Ivan Niven's *Maxima and Minima Without Calculus* (MAA Dolciani Series, No. 6 (1981), 7–11], for example, the derived formula for the area of a quadrilateral with sides a, b, c, d is

$$A^2=(s-a)(s-b)(s-c)(s-d)-\frac{abcd}{2}[1+\cos(\theta+\lambda)],$$

where $s=(a+b+c+d)/2$ and θ, λ are a pair of interior angles not sharing a common side. When the quadrilateral is inscribable in a circle, $\theta+\lambda=\pi$ and so we obtain Brahmagupta's formula

$$A=\sqrt{(s-a)(s-b)(s-c)(s-d)}\,.$$

———o———

Income Tax Averaging and Convexity

Michael Henry and G. E. Trapp, Jr., West Virginia University, Morgantown, WV

To compute the income tax for a given year, one method that can be used is income averaging. Roughly, this method is applicable when your taxable income for the current year is "much larger" than that for each of the four previous years. For instance, income averaging is often appropriate for college graduates who have completed the first year of their new job after having supported themselves with grant/scholarship money and part-time jobs during college.

To quote from the Internal Revenue Service Publication 17 [Your Federal Income Tax (For Individuals) for Use in Preparing 1982 Returns]: "If your income has increased substantially this year, it may be to your advantage to figure your tax under the income averaging method. Under this method, part of an unusually large amount of taxable income can be taxed at lower rates, and this may result in a lower tax for you."

But Publication 17 later warns that "... figuring your tax using the income averaging method may not give you a lower tax." Having read this caveat, one may wonder if the situation is, in fact, more complicated than presented. Could income averaging give a higher tax?

In this note, we use a simple convexity argument to show that higher taxes are not possible: *you can never lose money by averaging as opposed to computing the standard tax*. We will use the following notation:

B	The sum of your incomes for the four years immediately preceding this year's income.
I	This year's income.
A	The averageable income ($A = I - 0.3B$).
$T(x)$	The standard tax on income amount x.

Two coefficients are used below, 0.2 and 0.3. These coefficients have been used in all recent years for the averaging calculation, and are expected to be used in the 1983 tax year averaging calculation. The steps in computing the tax using income averaging are as follows (see Publication 17 for details):

$$\text{Find } T(0.2A + 0.3B) \tag{1}$$

$$\text{Find } T(0.2A + 0.3B) - T(0.3B) \tag{2}$$

$$\text{Compute } 4\left[T(0.2A + 0.3B) - T(0.3B)\right] + T(0.2A + 0.3B). \tag{3}$$

Simplifying and replacing A with $I - 0.3B$, we arrive at the following formula for the income-averaged tax:

$$\text{income-averaged tax} = 5T(0.2I + 0.24B) - 4T(0.3B). \tag{4}$$

The tax generated by (4) is the one we wish to compare with the standard tax $T(I)$. In particular, we claim that

$$T(I) \geqslant 5T(0.2I + 0.24B) - 4T(0.3B) \qquad \text{for all } I \text{ and } B. \tag{5}$$

We will establish (5) in a manner that will also help to shed some light on the roles played by the coefficients 0.2 and 0.3. To this end, replace 0.2 and 0.3 in equations (1) through (3) with u and v, respectively, and then simplify to obtain more general versions of (4) and (5).

$$T(I) \geqslant 5T\left[uI + (1-u)vB\right] - 4T(vB). \tag{5'}$$

Now recall that $T(x)$ is convex if for any $a \geqslant 0$ and $b \geqslant 0$ satisfying $a + b = 1$, there follows

$$aT(x_1) + bT(x_2) \geqslant T(ax_1 + bx_2). \tag{6}$$

It will shortly be demonstrated that the tax function $T(x)$ is convex. Based on this, we first rewrite (5′) as

$$.2T(I) + .8T(vB) \geqslant T\left[uI + (1-u)vB\right]. \tag{7}$$

For $u = .2$, we see that (7) is a special case of (6). Thus (7), and equivalently (5), hold true. This shows that one is *never* penalized for income averaging provided that T is convex (and that the IRS chooses $u = 0.2$). Note also that this argument is independent of the value assigned to v.

Now, from the tax tables in Publication 17, we see that T is of the form

$$T(x) = a_i + r_i(x - b_i) \qquad \text{for} \quad b_i \leqslant x < b_{i+1},$$

where

$$r_{i+1} > r_i \quad \text{and} \quad a_i + r_i(b_{i+1} - b_i) = a_{i+1}.$$

Therefore, T is a continuous and piecewise linear increasing function of x. Accordingly, T is indeed convex.

The choice $u = 0.2$ allows us to apply the convexity argument above. We now show that the choice of v influences the savings, if any, by income averaging. In particular, we will show that, as v increases, the tax payer's savings decrease. To this end, consider

$$D(I,B,v) = T(I) - 5T(.2I + .8vB) + 4T(vB),$$

the difference between $T(I)$ and the income-averaged tax. Now hold constant I and B, and take the partial derivative of D with respect to v, obtaining,

$$\frac{\partial D}{\partial v} = 4B\left[T'(vB) - T'(.2I + .8vB)\right]. \tag{8}$$

We claim that $\partial D/\partial v \leqslant 0$. Therefore, for a given I and B, we find that D is a decreasing function of v. To see this, we require two facts. First, since $T(x)$ is a piecewise linear function with increasing slopes, $T'(x)$ is a nondecreasing piecewise constant function (these derivative values or slopes are termed the marginal tax rates). Second, Publication 17 requires that $I - vB \geqslant 0$ before averaging may be used (in fact, the requirement is even more stringent: $I - vB$ must be greater than 3000). Therefore, $I \geqslant vB$ or $.2I + .8vB > vB$. Combining these two facts yields

$$T'(.2I + .8vB) \geqslant T'(vB).$$

Thus, from (8), we have $\partial D/\partial v \leqslant 0$. And this shows that tax savings from averaging decreases as v increases.

Since the term "averaging" is used, one might expect that v would be .25, since four prior years' incomes are "averaged." We do not know why $v = .3$ was chosen; but as we have seen, this choice of v results in less of a savings for the taxpayer.

We close by noting that if a flat tax rate (i.e., a simple linear tax function $T(x) = rx$) is ever adopted, then the averaged tax will be the same as the standard tax. The coefficient r is the *flat rate* which applies to any income x. The averaged tax and the standard tax would be the same because the convexity inequality in (6) would become an equality when the tax rate is given by $T(x) = rx$.