To generalize these results, we seek positive integers x that satisfy

$$x^2 + (x+1)^2 + (x+2)^2 + \cdots + (x+n)^2$$
$$= (x+n+1)^2 + (x+n+2)^2 + \cdots + (x+2n)^2, \qquad (3)$$

where n is a natural number. Simplifying (3) gives

$$x^2 - 2n^2x - n^2(2n+1) = 0,$$

the roots of which are

$$x = n(2n+1) \quad \text{and} \quad x = -n.$$

For $n = 1$ and $n = 2$, we have the systems $\{3,4;5\}$ and $\{10,11,12;13,14\}$, respectively. For $n = 3$, we obtain $\{21,22,23,24;25,26,27\}$ since

$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2.$$

On the Sphere and Cylinder

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Archimedes was so fond of his discovery, *that the ratio of the volume of a sphere to that of its circumscribing cylinder is the same as the ratio of their respective surface areas*, that he requested a sphere with its circumscribing cylinder be engraved on his tombstone after he died. Archimedes' result tells us that the surface area of a sphere is the same as the *lateral* surface area of its circumscribing cylinder. Actually, the surface area of any band on the sphere cut off by two parallel planes is the same as that of the projected band on its circumscribing cylinder. This beautiful fact was demonstrated by Arthur Segal in his Classroom Capsule "A Note on the Surface of a Sphere" [TYCMJ 13 (January 1982), 63–64]. Our objective is to view this result in n-dimensional settings where some interesting comparisons and constrasts with the 2- and 3-dimensional cases are possible.

Let $B^n(R) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leqq R^2\}$ be the n-dimensional (closed) ball with radius R and centre O, and let $S^{n-1}(R) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = R^2\}$ be the $(n-1)$-dimensional sphere (sitting in $\mathbb{R}^n$), which is the boundary of $B^n(R)$. We let $v\{B^n(R)\}$ and $v\{S^n(R)\}$ denote the volumes of $B^n(R)$ and $S^n(R)$, respectively. Note, for example, that $v\{B^2(R)\}$ is the area of the "circle" with radius R and $v\{S^1(R)\}$ is its circumference. Also, $v\{B^3(R)\}$ is the volume of the "sphere" with radius R and $v\{S^2(R)\}$ is its surface area. It is reasonable to expect that $v\{B^n(R)\}$ is proportional to R^n, and that $v\{S^{n-1}(R)\}$ is proportional to R^{n-1}. Hence, it is plausible that $v\{S^{n-1}(R)\}$ is equal to $2R$ times $v\{S^{n-2}(R)\}$ times a constant k_n which may depend on n. When $n = 3$, this says that the surface area of a sphere is equal to k_3 times the lateral surface area of its circumscribing cylinder; and from what has been said, k_3 should be 1. This leads one to query: what is k_n in general?

Let us first relate $v\{B^n(R)\}$ with $v\{S^{n-1}(R)\}$. Heuristically speaking, the volume of a very thin "shell" of thickness e is given by $v\{B^n(R)\} - v\{B^n(R-e)\}$ on the one hand, and roughly by $ev\{S^{n-1}(R)\}$ on the other. Writing $v\{B^n(R)\} = cR^n$, we obtain roughly

$$ev\{S^{n-1}(R)\} = cR^n - c(R-e)^n$$

$$= cnR^{n-1}e + \text{terms involving } e^2, e^3, \text{etc.}$$

Letting e become very very small, we can say that $v\{S^{n-1}(R)\} = ncR^{n-1}$. Hence, we expect to have

$$Rv\{S^{n-1}(R)\} = nv\{B^n(R)\}. \tag{1}$$

Formula (1), obtained heuristically, can be formally verified based on the fact [M. Fraser, "The Grazing Goat in n Dimensions," CMJ 15 (March 1984), p. 129] that

$$v\{B^n(R)\} = \begin{cases} \dfrac{\pi^{n/2}}{(n/2)!} R^n, & n \text{ even} \\ \dfrac{2^{(n+1)/2}\pi^{(n-1)/2}}{n(n-2)\cdots 3\cdot 1} R^n, & n \text{ odd.} \end{cases} \tag{2}$$

(Another verification at a slightly more advanced level is to apply the n-dimensional version of the Divergence Theorem to the function $F(x_1, \ldots, x_n) = (x_1, \ldots, x_n)$.)

Substituting (1) into our assumed relation

$$v\{S^{n-1}(R)\} = 2Rk_n v\{S^{n-2}(R)\},$$

we obtain

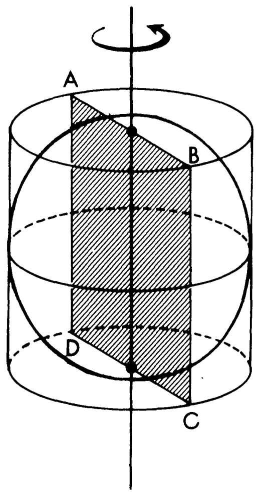
$$k_n = \frac{n \cdot v\{B^n(R)\}}{2R(n-1)\cdot v\{B^{n-1}(R)\}}. \tag{3}$$

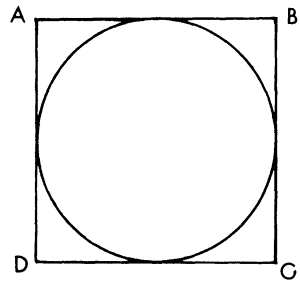
Thus, k_n is readily calculated via (2). Since k_3 turns out to be 1, we see that $v\{S^2(R)\}$ (the surface area of the sphere in $\mathbb{R}^3$) equals $2R$ times $v\{S^1(R)\}$ (the circumference of its circumscribing cylinder). In general, $k_n \neq 1$ so that the surface area of a "sphere" in higher dimension is *no longer* the same as the lateral surface area of its "circumscribing cylinder."

However, the full beauty of this generalization can be seen by comparing it with Archimedes' discovery. Using (1) and (3), we obtain

$$\frac{v\{S^{n-1}(R)\}}{\left[2Rv\{S^{n-2}(R)\} + 2v\{B^{n-1}(R)\}\right]} = \frac{n-1}{n}k_n = \frac{v\{B^n(R)\}}{2Rv\{B^{n-1}(R)\}}. \tag{4}$$

The third ratio in (4) is the volume of the "sphere" to that of its "circumscribing cylinder" (defined as $[-R, R] \times B^{n-1}(R)$). The first ratio is the surface area of the "sphere" to that of its "circumscribing cylinder" (having 'lateral' surface area $2R \cdot v\{S^{n-2}(R)\}$ and areas $v\{B^{n-1}(R)\}$ for its top and its bottom). Archimedes discovered the particular case $n = 3$, where the ratio is $\frac{2}{3}$. One can explain (4) heuristically by revolving an $(n-1)$-dimensional "ball" and its "circumscribing cylinder" about an axis. Let us illustrate this with Archimedes' case $n = 3$.





Think of the sphere as made up of infinitely many copies of the circle, and the cylinder as made up of infinitely many copies of the square. For each cross section (which is actually the 2-dimensional case where $k_2 = \pi/2$ in (4)),

$$\frac{\text{area of circle}}{\text{area of square}} = \frac{\text{circumference of circle}}{\text{perimeter of square}} \ (= \pi/4).$$

Hence, heuristically speaking, we have in the 3-dimensional case

$$\frac{\text{volume of sphere}}{\text{volume of cylinder}} = \frac{\text{surface area of sphere}}{\text{surface area of cylinder}}.$$

This is the result of which Archimedes was so fond.

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Distance From a Point to a Line

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The figure below can be used to obtain a simple derivation of the formula for the distance between a point $P(a, b)$ and the line $Ax + By + C = 0$.