

References

1. I. G. Bashmakova and G. S. Smirnova, The birth of literal algebra, *American Mathematical Monthly* 106:1 (1999) 57–66.
2. H. S. M. Coxeter, *The Real Projective Plane*, 3rd ed., Springer-Verlag, 1993.
3. G. Hessenberg, Über einen geometrischen Calcul, *Acta Mathematica*, Vol. 29 (1905).
4. Guido Lasters and David Sharpe, From Pascal to groups, *Mathematical Spectrum*, 29:3 (1996/7) 51–53.
5. K. G. C. von Staudt, *Beiträge zur Geometrie der Lage*, Nuremberg, 1857.
6. O. Veblen and J. W. Young, *Projective Geometry*, Vol. I, Ginn and Company, 1910.

Integrals of $\cos^{2n}x$ and $\sin^{2n}x$

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Integrals of the even powers of sine and cosine are notoriously difficult, and most texts approach them either by half-angle identities for $\cos^2 x$ and $\sin^2 x$ or by using reduction formulas. There is also a nice application of complex numbers that allows closed formulas to be derived quite easily. It is based on DeMoivre's formula

$$z^n = r^n(\cos n\theta + i \sin n\theta). \quad (1)$$

Let

$$z = \cos x + i \sin x,$$

so

$$\frac{1}{z} = \cos x - i \sin x$$

and, therefore,

$$\cos x = \frac{1}{2} \left(z + \frac{1}{z} \right) \quad \text{and} \quad \sin x = \frac{1}{2i} \left(z - \frac{1}{z} \right). \quad (2)$$

Applying the binomial formula to (2) gives

$$\cos^{2n} x = \frac{1}{4^n} \sum_{k=0}^{2n} \binom{2n}{k} z^{2n-2k} \quad (3)$$

and

$$\sin^{2n} x = \frac{1}{4^n} \sum_{k=0}^{2n} \binom{2n}{k} (-1)^{n-k} z^{2n-2k}. \quad (4)$$

From DeMoivre's formula (1) it follows that

$$z^{2n-2k} = \cos(2n-2k)x + i \sin(2n-2k)x,$$

which transforms (3) and (4) into

$$\cos^{2n} x = \frac{1}{4^n} \sum_{k=0}^{2n} \binom{2n}{k} \cos(2n-2k)x + \frac{i}{4^n} \sum_{k=0}^{2n} \binom{2n}{k} \sin(2n-2k)x$$

and

$$\begin{aligned}\sin^{2n}x &= \frac{1}{4^n} \sum_{k=0}^{2n} \binom{2n}{k} (-1)^{n-k} \cos(2n-2k)x \\ &\quad + \frac{i}{4^n} \sum_{k=0}^{2n} \binom{2n}{k} (-1)^{n-k} \sin(2n-2k)x.\end{aligned}$$

The imaginary parts in these expressions must be zero since $\cos^{2n}x$ and $\sin^{2n}x$ are real. Also,

$$\binom{2n}{m} \cos(2n-2m)x + \binom{2n}{2n-m} \cos(2m-2n)x = 2 \binom{2n}{m} \cos(2n-2m)x$$

and $\cos(2n-2n)x = 1$. Hence,

$$\cos^{2n}x = \frac{1}{4^n} \left[\binom{2n}{n} + 2 \sum_{k=0}^{n-1} \binom{2n}{k} \cos(2n-2k)x \right]$$

and

$$\sin^{2n}x = \frac{1}{4^n} \left[\binom{2n}{n} + 2 \sum_{k=0}^{n-1} \binom{2n}{k} (-1)^{n-k} \cos(2n-2k)x \right].$$

The substitution $j = n - k$ changes these formulas to

$$\cos^{2n}x = \frac{1}{4^n} \left[\binom{2n}{n} + 2 \sum_{j=1}^n \binom{2n}{n-j} \cos 2jx \right] \quad (5)$$

and

$$\sin^{2n}x = \frac{1}{4^n} \left[\binom{2n}{n} + 2 \sum_{j=1}^n \binom{2n}{n-j} (-1)^j \cos 2jx \right]. \quad (6)$$

From here,

$$\int \cos^{2n}x dx = \frac{1}{4^n} \left[\binom{2n}{n} x + \sum_{j=1}^n \frac{1}{j} \binom{2n}{n-j} \sin 2jx \right] + C \quad (7)$$

and

$$\int \sin^{2n}x dx = \frac{1}{4^n} \left[\binom{2n}{n} x + \sum_{j=1}^n \frac{(-1)^j}{j} \binom{2n}{n-j} \sin 2jx \right] + C. \quad (8)$$

It remains to make a few remarks. First, (7) and (8) at once lead to Wallis' formulas

$$\int_0^{\pi/2} \cos^{2n}x dx = \frac{1}{4^n} \binom{2n}{n} \frac{\pi}{2} \quad \text{and} \quad \int_0^{\pi/2} \sin^{2n}x dx = \frac{1}{4^n} \binom{2n}{n} \frac{\pi}{2}.$$

Second, the integrals of $\cos^{2n}x$ and $\sin^{2n}x$ can be easily obtained from each other by the substitution $x = \frac{\pi}{2} - \theta$. Formulas (5) and (6) also offer a glimpse into the subject of trigonometric polynomials and Fourier series. Furthermore, the difficult integral of $(x^2 + 1)^{-n-1}$ is reduced by the substitution $x = \tan \theta$ to the integral of $\cos^{2n}\theta$. Finally, (5) and (6) can be used to find the Laplace transforms of $\cos^{2n}x$ and $\sin^{2n}x$.